Fourier Oracles for Computer-Aided Improvisation

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Abstract

The mathematical study of the diatonic and chromatic universes in the tradition of David Lewin (9) and John Clough (6) is a point of departure for several recent investigations. Surprisingly, Lewin's original idea to apply finite Fourier transform to musical structures has not been further investigated for four decades. It turns out that several music-theoretically interesting properties of certain types of musical structures, as the partial symmetry of Fourier balances, maximal evenness (5) and well-formedness (4), allow alternative characterizations in terms of their Fourier transforms. The paper explores two particularly interesting cases: vanishing Fourier coefficients as an expression of chord symmetry and maximal Fourier coefficients as a reinterpretation of maximally even scales. In order to experimentally explore the Fourier approach we design an interactive playground for rhythmic loops. We propose a Fourier-based approach to be integrated as an "Scratching"-interface in the OMAX environment (built on OpenMusic and MaxMSP) which allows to interactively change a rhythm through a gestural control of its Fourier image. A collection of theoretical tools in $OpenMusic$ visual programming language helps the improviser to explore some new musical situations by inspecting mathematical and visual characteristics of the Fourier image.

1 David Lewin's call for Fourier

In a few lines at the end of his very first paper (9), David Lewin tantalizingly alludes to Fourier transforms and convolution products to explain how he was led to the special symmetries he considers in his paper about the relative intervallic content of two chords.

In the present paper we elaborate some of Lewin's ideas by applying the Finite Fourier Transform (FT) to more several of musical structures. To begin with, we model the set of notes modulo octave by $\mathbf{Z}_c$ ($c = 12$ in the equal division of the octave in twelve parts) and define, in this section, the Fourier transform of a subset A as a FT of its characteristic function:

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

The usual properties of FT apply and are indeed elementary in this simple discrete case. It is worthwhile to note that the usual musical operations (transposition, retrogradation, even complementation) do not change the modulus (i.e. length) of the FT: this means already that $|\mathcal{F}_A|$ is a good musical invariant.

The most interesting property is that the FT of the relative intervallic content of two chords (a function stating the number of occurences of any interval between A, B) is simply the product of their respective Fourier transforms, as FT turns convolution product into ordinary product:

$$\mathcal{F}(IC(A,B)) = \mathcal{F}_A \times \mathcal{F}_{-B}$$

Interestingly, this line of thought leads to a "one line" proof of Babbitt's hexacord theorem on which Babbitt and Lewin were both working hard at the same time. It was also the basis of Dan Tudor Vuza's original work (13) on 'Vuza canons' which has been also implemented in *OpenMusic*.

1.1 Vanishing Fourier Transform of Characteristic Functions

Lewin's interest though was in a kind of inverse problem, reconstructing (say) A from B and the intervallic content. This is possible when $\mathcal{F}_{-B}$ is non vanishing, and this excludes precisely the five cases that Lewin put forward as 'augmented-triad property' and the like in (9), and simply 'FOURPROP(i)' in (10). See (12), chap. 3 for more details. Of interest to us is the fact that $\mathcal{F}_{-B}(k)$ will vanish essentially when set B exhibits a notion akin to periodicity, the **Fourier balance**. For instance the set $B = (0, 4, 5)$, like the augmented triad $(0, 4, 8)$, has FOURPROP(4), as it is evenly distributed among the three diminished seventh chords. Suffice to say here that the Fourier transform enables to give an interesting generalization of periodic subsets (or Messiaen transposition limited modes) of $\mathbf{Z}_c$.

1.2 Maximal Fourier Coefficients and Maximal Evenness

On the other hand, when the FT is as far as possible from vanishing (for some value k of its argument), it must characterize other interesting subsets of $\mathbf{Z}_c$. As noticed in (12), and later formally proved by one of the authors, this case corresponds to the well known family of Maximally Even Sets (ME sets), as defined in the general case by (5).

Theorem 1 *A subset $A \in \mathbf{Z}_c$ of cardinality d is **Maximally Even** if $|\mathcal{F}_A(d)| \geq |\mathcal{F}_B(d)|$ for all subsets B of cardinality d.*

Intuitively, this means that A approximates as well as possible for a discrete subset an even division of cardinality c by d terms. For instance, in the simplest case when d is a divisor of c, and $A = \{0, c/d, 2c/d, \}$

$$\mathcal{F}_A(d) = \sum_{k \in A} e^{2i\pi kd/c} = 1 + 1 + \ldots = d$$

which is clearly maximal for a sum of d terms, each of which has length 1. In less clear-cut cases, like the major scale ($c = 12, d = 7$), the maximum is strictly less than d.

Aside from characteristic functions there is another simple way to encode scales and periodic rhythms; namely as points on the unit circle. In this approach we do not need a chromatic universe where the scales are embedded. The notion of maximal evenness can be defined relative to any collection of N-note scales. A scale $X = \{e^{2\pi i t_1}, ..., e^{2\pi i t_N}\}$ will be called **Maximally Even** among all scales within a given family $\mathfrak{S}$ if its Fourier coefficient a_1 has maximal radius among the corresponding Fourier coefficients a_1' for all members of the family $\mathfrak{S}$.

This recovers the ordinary definition of maximally evenness, where the $\mathfrak{S}$ consists of all N-note subscales of a equally distributed chromatic universe. Scales in step order, whose zero'th Fourier coefficient a_0 is bigger than a_1 can be characterised as clusters.

The unit-circle representation of scales is the point of departure for our modeling of periodic rhythms in the following section.

2 Fourier "Scratching"

Musical Rhythms are often described as sequences of inter-onset intervals or simply as sets of musical onsets. The former manner of description focusses on the internal connectivity of a rhythm while the latter one conceives rhythms as sets of events. The sequencial temporal order of the single events is then implicitely given by the order of the linear order on the onset axis. The following model introduces a concept of sequentiality independently from the temporal order. Identity of temporal and sequential order serves only as a point of departure for the exploration of a variety of other kinds of solidarity between the two. For example, think of the linear order of musical pitch. An ascending scale exemplifies a strict solidarity between both orders. A systematic exchange of each second note with its predecessor destroys this strikt solidarity, but still exemplifies a generalized solidarity principle. The manipulation of the finite Fourier transforms of the rhythms thereby allows to intuively perform slight global changes on the rhythms without changing each single event "by hand".

2.1 Rhythmic Loops

In this application we assume a cyclic organisation of musical time with a fixed abstract period $d = 1$. Practically, the actual interpretation of this abstract period (in terms of physical time in a performance) might vary, but we do not elaborate this further. Let $\mathbb{U} = \{\phi_t = e^{2\pi i t} \,|\, 0 \leq t < 1\}$ denote the unit circle and let $\mathbb{U}_N = \{\phi_{k/N} = e^{2\pi i k/N} \,|\, 0 \leq k < N\}$ denote the discrete subcircles of cardinality N (i.e. N'th roots of unity). Of all the parameters which might be postulated for the specification of a musical event we isolate just the following two: *phase* $\phi_t \in \mathbb{U}$ and *intensity* $r \geq 0$ ($r \in \mathbb{R}$). For all additional parameters we postulate a musical space $\mathbb{S}$. The selected parameters phase and intensity may vary independently with one exception: vanishing intensity implies zero phase. This trick allows us to identify musical events with pairs $(z, s) \in \mathbb{C} \times \mathbb{S}$. Intensity and phase are thus jointly encoded in terms complex numbers $z = r\phi_t$.

The musical space $\mathbb{S}$ may encode several aspects such as pitch, timbre, direction (orthophony). In order to motivate the idea of rhythmic loops we assume a modality for creating loops (closed curves) within this space $\lambda : \mathbb{U} \to \mathbb{S}$ or discrete loops $\lambda_N : \mathbb{U}_N \to \mathbb{S}$. For simplicity we assume that all the discrete loops λ_N are restrictions of a previously chosen continuous loop λ. As in the case of the period d this loop may vary within the space S, but we do not elaborate on that here. By definition, we conceive a rhythmic N-loop as a map

$$\rho \times \lambda_N : \mathbb{U}_N \to \mathbb{C} \times \mathbb{S}.$$

In the sequel we focus on the purely rhythmic part of this map, namely $\rho : \mathbb{U}_N \to \mathbb{C}$. However, it is important to notice that the musical motivation for this is encoded in the other map $\lambda_N : \mathbb{U}_N\mathbb{S}$. which embodies a sequentiality with regard to the space S. We claim that the manipulation of Fourier Coefficients of ρ results in a manipulation of the musical solidarity between sequentiality in S on the one hand and cyclic time on the other.

2.2 Analysis and Resynthesis of Rhythms

The application of finite Fourier transform is inspired by the surprising success of this method in sound processing and works more or less straight forward. However, finite Fourier transform serves in signal processing as a approximation of the continuous case. The present application to musical rhythm — such as the applications to Chords and Scales mentioned previously — is based on finite Fourier transform as an autonomous analogue to the continous case. The finite Fourier transform is a linear bijection of $\mathbb{C}$

2.3 Dragging, Eliminating and Inserting of Fourier Coefficients

The section title "Fourier Scratching" refers to a manipulation of rhythms through gestural distortions of their Fourier images. The "Fourier DJ" grabs a Fourier coefficient and moves it in order to modify the entire rhythm in a subtle way. The theory guarantees the existence of the inverse transform. Of course, the musical interpretation depends on a suitably chosen quantification of the musical time and intensity. Figure 1 shows an example for a series of progressive rhythm deformations.

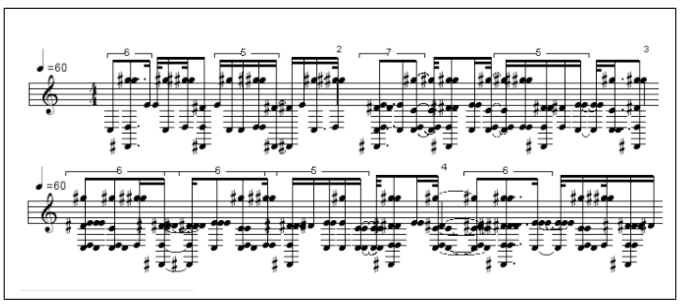

Figure 1: The OpenMusic Voice object displays a series of progressive deformations of an initial rhythmic pattern

Aside from a purely experimental exploration of the musical effects like in the previous example, we propose some theoretical feedback to the improviser (see next section).

The number of events per rhythm does nod need to be fixed. Our postulate that all discrete loops $\lambda_N : \mathbb{U}_N \to \mathbb{S}$ are restrictions of a continuous loop $\lambda : \mathbb{U} \to \mathbb{S}$ allows to let the rhythmic loop to vary in cardinality. Filter techniques as well es zero-padding offer modalities for contiguous transitions between rhythms of different cardinality.

3 Towards a Guided Improvisation in the OMAX Environment

Computer-Aided approaches to musical improvisation need not to be restricted to gestural interaction and sound processing. We would like to support the challenging idea of using theory-oriented tools, enhancing the improvisers' conscious knowledge about their own creative activity. In this section we describe a concrete instance of an architecture which combines the realtime capacities of MAX/MSP for live performance with the exploratory power of OpenMusic visual programming language. This architecture, which has been proposed by (1), allows the combination of realtime interaction with high level operations such as the inspection of musical properties of the ongoing process. Fourier transforms on rhythms and other musical objects provide a particularly interesting case study, as these give instantaneous snapshots of the global tendency of the musical process. Aside from merely inspecting the actual Fourier transform; the environment offers a the interactive comparison of an actual diagnosis with families of previously investigated musical situations. Three simple examples may illustrate the general philosophy of such an interaction.

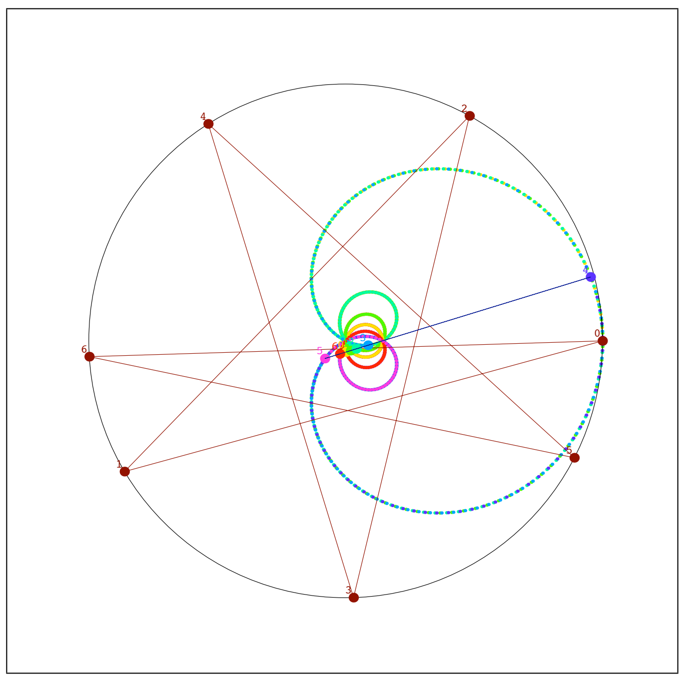

Figure 2: Localisation of the Fourier coefficients of the diatonic scale on the characteristic curve for generated 7-note scales

1. Suppose the improvisor attempts to turn his actual rhythmic pattern into a palindromic one. While the visual inspection of the rhythm itself may be as complicated as listening to it, the Fourier transforms of palindromes correspond to alignments of the coefficients along lines. The best linear fit to the actual Fourier image can serve as an oracle for a path toward the desired palindromic state.

2. Wellformed rhythms, i.e. the rhythmic analogues to wellformed scales, have the property of being being palindromic and as well as having their Fouriercoeffients positioned on a characteristic curve. This curve is the balanced sum of the first N (complex) partials, where N denotes the number of beats and parametrizes the Fourier-images of the generation-order encoding all generated N-rhythms. Wellformed rhythms keep this alignment also in sequential-order encoding (see Figure 2).

3. The improviser may find his actual music particularly fascinating without understanding it in the live situation. A Fourier snapshot can offer him a characteristic picture, which can be interactively manipulated (see the previous section) or saved as a potential component of an oracle.

Although the full integration of this playground in OMAX is still under development we can alredy present some instructive OpenMusic implementations.

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