ON SOME PROPERTIES OF PERIODIC SEQUENCES IN ANATOL VIERU’S MODAL THEORY

MORENO ANDREATTA — DAN T. VUZA

ABSTRACT. Algebraic methods have been currently applied to music in the second half of the twentieth-century (see [M. Andreatta: Group-theoretical Methods applied to Music, unpublished dissertation, 1997], [M. Chemilier: Structure et Méthode algébriques en informatique Musicale. Thèse de doctorat, L. I. T. P., Institut Blaise Pascal, 1990] and [G. Mazzola et al.: The Topos of Music—Geometric Logic of Concepts, Theory and Performance] for main references]. By starting from Anatol Vieru’s compositional technique based on finite difference calculus on periodic modal sequences, as it has been introduced in his book [Cartea modurilor, 1 (Le livre des modes, 1). Ed. Muzicala, Bucarest, 1980. Revised ed. The book of modes, 1993], the present essay tries to generalize some properties by means of abstract group theory. Two main classes of periodic sequences are considered: reducible and reproducible sequences, replacing respectively Vieru’s modal and irreducible sequences. It turns out that any periodic sequence can be decomposed in a unique way into a reducible and a reproducible component.

1. Reducible and reproducible sequences

For any sequence \( f \) defined on \( \mathbb{Z} \) taking values in a finite abelian group \( G \) we define the translated sequence \( Tf \) and the sequence of differences \( Df \) by

\[
Tf(x) = f(x + 1), \quad Df(x) = f(x + 1) - f(x).
\]

The relationship between the translated sequence and the sequence of differences is expressed by the following equation:

\[
D = T - 1.
\]
**Definition 1.** The sequence $f$ is called $m$-periodic if $f(x + m) = f(x)$ for any $x \in \mathbb{Z}$.

Affirming that $f$ is $m$-periodic, it is equivalent to the relation $T^m f = f$. If $f$ is $m$-periodic, then $Tf$ and $Df$ are also $m$-periodic.

**Definition 2.** The sequence $f$ is called reducible if an integer $k \geq 0$ does exist such that $D^k f = 0$.

The sequence $f$ is called reproducible if an integer $k \geq 0$ does exist such that $D^k f = f$.

By $\text{Red}(G)$ and $\text{Rep}(G)$ we will designate respectively the set of reducible and reproducible sequences taking values in $G$ (also called $G$-valued sequences).

**Example 1.** Example of a reducible sequence. In Anatol Vieru’s book [10] a $\mathbb{Z}_{12}$-valued sequence of period 72 is considered. It can be represented by means of 6 lines of 12 elements

\[
\begin{array}{cccccccccccc}
4 & 1 & 0 & 8 & 8 & 7 & 0 & 6 & 8 & 1 & 4 & 0 \\
8 & 11 & 4 & 6 & 0 & 5 & 4 & 4 & 0 & 11 & 8 & 10 \\
0 & 9 & 8 & 4 & 4 & 3 & 8 & 2 & 4 & 9 & 0 & 8 \\
4 & 7 & 0 & 2 & 8 & 1 & 0 & 0 & 8 & 7 & 4 & 6 \\
8 & 5 & 4 & 0 & 0 & 11 & 4 & 10 & 0 & 5 & 8 & 4 \\
0 & 3 & 8 & 10 & 4 & 9 & 8 & 8 & 4 & 3 & 0 & 2 \\
\end{array}
\]

Any column is a periodic sequence obtained by adding modulo 12 a constant value to a basis element (i.e., the top of the column). By putting the constant value as an index for the basis element, one has the following compact expression

\[
D^1 = (2_6 9_6 11_6 8_6 0_6 11_6 5_6 6_6 2_6 5_6 3_6 8_6),
\]

\[
D^2 = (0 7 2 9 4 11 6 1 8 3 1 0 5),
\]

\[
D^3 = (7),
\]

\[
D^4 = 0.
\]

**Example 2.** Example of a reproducible sequence

\[
f = (8 11 0 1 4 0),
\]

\[
D^1 = (3 1 1 3 8 8),
\]

\[
D^2 = (10 0 2 5 0 7),
\]

\[
D^3 = (2 2 3 7 7 3),
\]

\[
D^4 = f.
\]
Theorem 1. Let \( d_i, 0 \leq i \leq N \), be some integers such that at least one of them is relatively prime to the number of elements of \( G \). Then, an integer \( m \) does exist such that any sequence verifying

\[
\sum_{i=0}^{N} d_i T^i f = 0
\]

is \( m \)-periodic.

Corollary 2. Reducibles and reproducibles sequences are periodic.

Theorem 3. All periodic sequence can be decomposed in a unique way

\[
f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}(G), \quad f_{\text{rep}} \in \text{Rep}(G).
\]

Proof. Let \( f \) be \( m \)-periodic. Being the collection of sequences \( m \)-periodic finite, two integers \( k, l \geq 1 \) do exist such that \( D^k f = D^{k+l} f \). By induction on \( r \) one has \( D^{rl} f = D^{k+rl} f \). In the same way it can be shown that two integers \( r, s \geq 1 \) may exist such that \( D^{rl} f = D^{(r+s)l} f \). We put

\[
f_{\text{red}} = f - D^{rl} f, \quad f_{\text{rep}} = D^{rl} f.
\]

It follows that \( D^k (f - D^{rl} f) = 0 \), \( D^{sl} D^{rl} f = D^{rl} f \), which means that \( f_{\text{red}} \) and \( f_{\text{rep}} \) give the needed decomposition.

The unicity comes from the relation \( \text{Red}(G) \cap \text{Rep}(G) = \{0\} \). \( \square \)

2. Decomposition of \( \mathbb{Z}_n \) into \( p \)-groups

There is a one-to-one relation between the subgroups of \( \mathbb{Z}_n \) and the family of integers \( d \) that divide \( n \) by \( 1 \leq d \leq n \), i.e., for any such \( d \) we may take the unique subgroup of \( \mathbb{Z}_n \) with \( d \) elements. The latter can be characterized as the set of \( z \in \mathbb{Z}_n \) such that \( dz = 0 \) or, equivalently, as the set \( \frac{n}{d} \mathbb{Z}_n \) of elements having the form \( \frac{n}{d} z \) where \( z \) belongs to \( \mathbb{Z}_n \).

Definition 3. The abelian group \( G \) is a direct sum of a family of subgroups \( G_1, \ldots, G_m \) of \( G \) if any \( x \in G \) may be decomposed in a unique way into a sum \( x_1 + \cdots + x_m \) with \( x_i \in G_i \) for \( 1 \leq i \leq m \).

We will put \( G = \bigoplus_{i=1}^{m} G_i \).

Definition 4. Let \( p \) be a prime number. A finite abelian group is called \( p \)-group if its cardinality is a power of \( p \).
**Theorem 4.** Any group $\mathbb{Z}_n$ is a direct sum of its $p$-maximals subgroups.

If $n = \prod_{i=1}^{m} p^{k_i}$ is the decomposition of $n$ into prime factors, the decomposition of $\mathbb{Z}_n$ in maximal $p$-subgroups can be written as follows

$$\mathbb{Z}_n = \bigoplus_{i=1}^{m} G_{p^{k_i}},$$

where $G_{p^{k_i}}$ is the subgroup of $\mathbb{Z}_n$ with $p^{k_i}$ elements. The decomposition of any $z \in \mathbb{Z}_n$ defines the elements $\pi_i(z) \in G_{p^{k_i}}$ in a unique way such that $z = \sum_{i=1}^{m} \pi_i(z)$. The arrows $\pi_i: \mathbb{Z}_n \rightarrow G_{p^{k_i}}$ are group morphisms.

The $p_i$-component $\pi_i(z)$ of $z$ is the unique element $y \in \mathbb{Z}_n$ satisfying the relations $p^{k_i}y = \frac{n}{p^{k_i}}(z - y) = 0$ in $\mathbb{Z}_n$. Let $q_i$ be an integer verifying

$$q_i \frac{n}{p^{k_i}} = 1 \mod p^{k_i}.$$

It follows that $p^{k_i} \frac{n}{p^{k_i}} q_i z = \frac{n}{p^{k_i}}(z - \frac{n}{p^{k_i}} q_i z) = 0$ in $\mathbb{Z}_n$, which means that

$$\pi_i(z) = \frac{n}{p^{k_i}} q_i z.$$

**Example 3.** Subgroups of $\mathbb{Z}_{12}$

- $G_1 \{0\}$,
- $G_2 \{0, 6\}$,
- $G_3 \{0, 4, 8\}$, maximal 3-group,
- $G_4 \{0, 3, 6, 9\}$, maximal 2-group,
- $G_6 \{0, 2, 4, 6, 8, 10\}$,
- $G_{12} \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

**Example 4.** Decomposition of $\mathbb{Z}_{12}$ into $p$-groups

$$\mathbb{Z}_{12} = G_3 \bigoplus G_4.$$

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Remark 1. Many theoretical works ([2], [8] and [10]) have already shown the musical interest of the decomposition of the cyclic group $\mathbb{Z}_{12}$ into a direct sum of its maximal $p$-groups from a theoretical, analytical and compositional point of view.

**Theorem 5.** Let $\varphi_i : \mathbb{Z}_n \to \mathbb{Z}_{p^i}$ be the canonic map. The map $\varphi : \mathbb{Z}_n \to \prod_{i=1}^m \mathbb{Z}_{p^i}$ defined by

$$\varphi(z) = (\varphi_1(z), \ldots, \varphi_m(z))$$

is a ring homomorphism with inverse given by

$$(z_1, \ldots, z_m) \mapsto \sum_{i=1}^m \frac{n}{p_i} q_i z_i.$$  

For any $\mathbb{Z}_n$-valued sequence $f$ we put $f_i(x) = \pi_i(f(x))$. We will say that $f = \sum_i f_i$ is the decomposition of $f$ corresponding to the decomposition of $\mathbb{Z}_n$ in $p$-groups.

### 3. Characterization of reducible sequences

**Proposition 6.** Let $f = \sum_j f_j$ be the decomposition of $f$ corresponding to the decomposition of $\mathbb{Z}_n$ in $p$-groups. Then $f \in \text{Red}(\mathbb{Z}_n)$ if and only if $f_j \in \text{Red}(\mathbb{Z}_n)$ for any $j$.

**Theorem 7.** Let $f$ be a $\mathbb{Z}_{p^k}$-valued sequence. Then $f$ is reducible if and only if it is $p^m$-periodic for a given $m \geq 0$.

**Proof.** By induction on $k$. For $k = 1$ the result comes from the following lemma.

\[ q_3 = 1, \quad 1 \cdot \frac{12}{3} = 1 \text{ mod } 3, \]

\[ q_4 = -1, \quad -1 \cdot \frac{12}{4} = 1 \text{ mod } 4, \]

$$\pi_3(x) = \frac{12}{3} q_3 x = 4x, \quad \pi_4(x) = \frac{12}{4} q_4 x = -3x.$$
Lemma 8. For any $\mathbb{Z}_p$-valued sequence $f$, one has
\[(T - 1)^{p^m} f = (T^{p^m} - 1) f.\]

Proof. By induction on $m$ one is conducted to the case $m = 1$.
\[(T - 1)^{p} f = \sum_{i=0}^{p} (-1)^{p-i} C_i^1 T^i f.\]

Since $p$ divides any $C_i^1$ by $1 \leq i \leq p - 1$ and since $p f = 0$ it follows that the sum of the right member is reduced to $T^p f - f$.

Let now the theorem be true for any $k_1 < k$. We designate $f_1$ the sequence defined by $f_1(x) = p f(x)$. It is clear that $f_1$ takes values in the subgroup $G_{p^{k-1}}$ with $p^{k-1}$ elements of $\mathbb{Z}_{p^{k-1}}$, isomorphic to $\mathbb{Z}_{p^{k-1}}$.

Let $f$ be $p^{m}$-periodic. Then $f_1$ is also $p^{m}$-periodic, i.e., reducible (induction hypothesis). Therefore, there exists $l \geq 1$ such that $D^l f_1 = 0$. By definition of $f_1$, it means that $p(D^l f)(x) = 0$ for any $x$, i.e., the sequence $D^l f$ takes values into the subgroup $G_p$ with $p$ elements of $\mathbb{Z}_p$, isomorphic to $\mathbb{Z}_p$. Since $D^l f$ is $p^{m}$-periodic, the induction hypothesis leads to the existence of $l_1 \geq 1$ such that $D^{l_1} D^l f = 0$, i.e., $D^{l_1 + l} f = 0$ and $f$ is reducible.

Let $f$ be reducible. Then $f_1 \in \text{Red}(G_{p^{k-1}})$ by induction hypothesis it follows that there exists $m_1 \geq 0$ such that $f_1$ is $p^{m_1}$-periodic. This means that $f(x) - f(x + p^{m_1}) \in G_p$ for any $x$. We define the sequence $f_2$ by $f_2(x) = f(x) - f(x + p^{m_1})$. Since $f_2 \in \text{Red}(G_p)$, the induction hypothesis guaranties the existence of $m_2 \geq 0$ such that $f_2$ is $p^{m_2}$-periodic. In short, $(1 - T^{p^{m_2}}) (1 - T^{p^{m_1}}) f = 0$. To conclude we just need the following.

Lemma 9. Let $f$ be a $\mathbb{Z}_n$-valued sequence such that $(1 - T^k)(1 - T^l) f = 0$. Then $f$ is $kln$-periodic.

Proof. From $(1 - T^i) f = T^k (1 - T^l) f$ one deduces that $(1 - T^i) f = T^{ki}(1 - T^l) f$ for any $i \geq 1$, and in particular that $(1 - T^{kl})(1 - T^l) f = 0$. By the same argument, $(1 - T^{kl}) f = T^{li}(1 - T^{kl}) f$ for any $i \geq 1$. This means that
\[(1 - T^{kln}) f = (1 - (T^{kl})^n) f = \left(\sum_{i=0}^{n-1} T^{kli}\right) (1 - T^{kl}) f = n (1 - T^{kl}) f = 0.\]

Corollary 10. $\text{Red}(\mathbb{Z}_n)$ is a ring.

Corollary 11. Let $f \in \text{Red}(\mathbb{Z}_n)$ be such that $f(x)$ is invertible in $\mathbb{Z}_n$ for all $x$. Then $f^{-1} \in \text{Red}(\mathbb{Z}_n)$.

Corollary 12. Let $k,l$ be two integers and let $g$ be defined by $g(x) = f(kx + l)$. If $f \in \text{Red}(\mathbb{Z}_n)$ then $g \in \text{Red}(\mathbb{Z}_n)$. 

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4. Characterization of reproducible sequences

**Proposition 13.** Let \( f = \sum_j f_j \) be the decomposition of \( f \) corresponding to the decomposition of \( \mathbb{Z}_n \) in \( p \)-groups. Then \( f \in \text{Rep}(\mathbb{Z}_n) \) if and only if \( f_j \in \text{Rep}(\mathbb{Z}_n) \) for any \( j \).

**Definition 5.** For every \( m \)-periodic sequence \( f \) and for every integer \( d \) dividing \( m \), we define the \( d \)-periodised obtained from \( f \) as the sequence \( \sum_{i=0}^{m/d-1} T^d f \).

The equation

\[
(1 - T^d) \sum_{i=0}^{m/d-1} (T^d)^i f = (1 - (T^d)^{m/d}) f = (1 - T^m) f = 0
\]

shows that the \( d \)-periodised sequence is indeed a \( d \)-periodic sequence.

**Theorem 14.** Let \( f \) be a \( \mathbb{Z}_{p^r} \)-valued \( m \)-periodic sequence. Then \( f \) is reproducible if and only if the \( p^r \)-periodised of \( f \) is zero, where \( p^r \) is the highest power of \( p \) that divides \( m \).

**Proof.** If \( f \) is reproducible, then its \( p^r \)-periodised is also reproducible by construction, but also reducible according to Theorem 7, because it is \( p^r \)-periodic. Then the \( p^r \)-periodised is zero, because \( \text{Rep} \cap \text{Red} = \{0\} \).

Conversely, by supposing that the \( p^r \)-periodised is zero, we may write the decomposition \( f = f_{\text{red}} + f_{\text{rep}} \), \( f_{\text{red}} \in \text{Red} \), \( f_{\text{rep}} \in \text{Rep} \).

According as we have shown, the \( p^r \)-periodised of \( f_{\text{rep}} \) is zero. Then the previous equation shows that the \( p^r \)-periodised of \( f_{\text{red}} \) must be zero too. Due to Theorem 7, \( f_{\text{red}} \) must be \( p^s \)-periodised for a given integer \( s \geq 0 \). Because of the fact that the \( m \)-periodised of \( f \) implies that of \( f_{\text{red}} \), the period of the latter divides \( m \) and \( p^s \), which shows that \( f_{\text{red}} \) is in fact \( p^r \)-periodic. As a consequence its \( p^r \)-periodised is equal to \( \frac{m}{p^r} f_{\text{red}} \). Since \( \frac{m}{p^r} \wedge p^k = 1 \), the multiplication by \( \frac{m}{p^r} \) is an automorphism of \( \mathbb{Z}_{p^k} \). From \( \frac{m}{p^r} f_{\text{red}} = 0 \) one may conclude that \( f_{\text{red}} = 0 \).

**Corollary 15.** Let \( f \) be a \( \mathbb{Z}_{p^k} \)-valued \( m \)-periodic sequence and let \( p^r \) be the highest power of \( p \) dividing \( m \). Then \( f \) is reproducible if and only if the relations

\[
\sum_{i=0}^{m/p^r-1} f(p^r i + x) = 0
\]

hold for all \( x \) such that \( 0 \leq x < p^r \).
5. Calculation of reducible and reproducible components of a periodic sequence

**Proposition 16.** Let $f = \sum_j f_j$ be the decomposition of the periodic sequence $f$ corresponding to the decomposition of $\mathbb{Z}_n$ in $p$-groups and let $f_j = f_{j,\text{red}} + f_{j,\text{rep}}$ be the decomposition of $f_j$ in a sum of a reducible and a reproducible sequence. Then the decomposition of $f$

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep},$$

is given by

$$f_{\text{red}} = \sum_j f_{j,\text{red}}, \quad f_{\text{rep}} = \sum_j f_{j,\text{rep}}.$$ 

**Theorem 17.** Let $f$ be a $\mathbb{Z}_{p^k}$-valued $m$-periodic sequence and let $f_{\text{per}}$ be the $p^r$-periodised of $f$, where $p^r$ is the highest power of $p$ dividing $m$. Let $\left(\frac{m}{p^r}\right)^{-1}$ be the inverse of $\frac{m}{p^r}$ mod $p^k$. Then the decomposition of $f$

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep},$$

is given by

$$f_{\text{red}} = \left(\frac{m}{p^r}\right)^{-1} f_{\text{per}}, \quad f_{\text{rep}} = f - \left(\frac{m}{p^r}\right)^{-1} f_{\text{per}}.$$ 

**Proof.** Since $f_{\text{red}}$ is $p^r$-periodic by construction, it is reducible according to the Theorem 7. The $p^r$-periodised of $f_{\text{rep}}$ is zero:

$$\sum_{i=0}^{m/p^r-1} T^{ip^r} f_{\text{rep}} = \sum_{i=0}^{m/p^r-1} T^{ip^r} f - \left(\frac{m}{p^r}\right) \left(\frac{m}{p^r}\right)^{-1} f_{\text{per}} = f_{\text{per}} - f_{\text{per}} = 0.$$ 

Due to the Theorem 14, $f_{\text{rep}}$ is reproducible. 

**Corollary 18.** Let $m$ and $n$ be two integers such that $m \land n = 1$ and let $f$ be a $\mathbb{Z}_n$-valued $m$-periodic sequence. $f$ is reproducible if and only if

$$\sum_{i=0}^{m-1} f(i) = 0.$$ 

6. Decomposition algorithm

1. Write the decomposition of $n$

$$n = \prod_{j=1}^{N} p_j^{k_j}$$

into prime factors.
2. Find the integers \( q_j \) such that
\[
q_j \frac{n}{p_j^{k_j}} = 1 \text{ mod } p_j^{k_j}.
\]

3. For all \( j \) build the sequences \( f_{j,\text{red}} \) and \( f_{j,\text{rep}} \) as follows.
4. Set \( f_j(x) = \varphi_j(f(x)) \), where \( \varphi_j : \mathbb{Z}_n \to \mathbb{Z}_{p_j^{k_j}} \) is the canonnic map.
5. Let \( p_j^{r_j} \) be the highest power of \( p_j \) dividing the period \( m \) of \( f \). Determine the inverse \( \left( \frac{m}{p_j^{k_j}} \right)^{-1} \) of \( \frac{m}{p_j^{k_j}} \text{ mod } p_j^{k_j} \).
6. Write down the \( m \) elements of the period of \( f_j \) as a table with \( \frac{m}{p_j^{r_j}} \) lines and \( p_j^{r_j} \) columns (if \( r_j = 0 \), the table will have an unique column). Add a line given by the elements of any column and by multiplying by \( \left( \frac{m}{p_j^{k_j}} \right)^{-1} \) module \( p_j^{k_j} \).
7. Let \( f_{j,\text{red}} \) be the \( p_j^{r_j} \)-periodic sequence defined by the line built at the previous step and build \( f_{j,\text{rep}} = f_j - f_{j,\text{red}} \).
8. By setting
\[
f_{\text{red}} = \sum_{i=1}^{N} q_j \frac{n}{p_j^{k_j}} f_{j,\text{red}}, \quad f_{\text{rep}} = \sum_{i=1}^{N} q_j \frac{n}{p_j^{k_j}} f_{j,\text{rep}},
\]
we have the decomposition of any periodic sequence in a reducible and reproducible component.

**Example 5.** Decomposition in reducible and reproducible components. Let \( f \) be the following \( \mathbb{Z}_{12} \)-valued sequence,
\[
f = (0 \ 0 \ 7 \ 7 \ 4 \ 4 \ 3 \ 4 \ 4 \ 7 \ 7).
\]
The sequences corresponding to the decomposition of \( \mathbb{Z}_{12} = G_3 \oplus G_4 \) are respectively
\[
f_1 = (0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1)
\]
\[
f_2 = (0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3)
\]

\[
\begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
2 & 2 & 1 \\
\div 4 \text{ mod } 3 & 2 & 2 & 1 \\
\times 4 \text{ mod } 12 & 8 & 8 & 4
\end{array}
\]
7. The sets of values of reducible and reproducible sequences

**Theorem 19.** Let \( \mathbb{Z}_n = \bigoplus_j G_j \) be the decomposition of \( \mathbb{Z}_n \) in \( p \)-groups. A subset \( M \) of \( \mathbb{Z}_n \) is the set of values of a reducible sequence if and only if two subsets \( M_j \subset G_j \) do exist such that \( M = \sum_j M_j \).

This theorem is a direct consequence of the following proposition.

**Proposition 20.** Let \( \mathbb{Z}_n = \bigoplus_j G_j \) be the decomposition of \( \mathbb{Z}_n \) in \( p \)-groups and let \( f = \sum_j f_j \) be the decomposition of the reducible sequence \( f \) corresponding to the decomposition of \( \mathbb{Z}_n \). Then \( f(\mathbb{Z}) = \sum_j f_j(\mathbb{Z}) \).

**Proof.** Clearly \( f(\mathbb{Z}) \subseteq \sum_j f_j(\mathbb{Z}) \). On the other hand, let \( y_j = f_j(x_j) \in f_j(\mathbb{Z}) \) and, for all integer \( p_j \) dividing \( n \), let \( p_j^r \) be the highest power of \( p_j \) dividing the period \( m \) of \( f \). According to the Theorem 7, \( f_j \) is \( p_j^r \)-periodic. A well known result of algebra gives the existence of \( x \in \mathbb{Z} \) verifying \( x = x_j \mod p_j^r \) for all \( j \). It follows that \( f(x) = f_j(x) \) for all \( j \), i.e.,

\[
\sum_j y_j = \sum_j f_j(x_j) = \sum_j f_j(x) = f(x) \in f(\mathbb{Z}).
\]

**Theorem 21.** For any subset \( M \subset \mathbb{Z}_n \) it does exist an element \( u \in \mathbb{Z}_n \) and a reproducible sequence \( f \) such that \( f(\mathbb{Z}) = u + M \).

**Proof.** It does exist an integer \( m \) such that \( m \wedge n = 1 \) and the \( m \)-periodic sequence \( g \) such that \( g(\mathbb{Z}) = M \). Since \( m \wedge n = 1 \), the inverse \( m^{-1} \) of \( m \) mod \( n \) does exist. We set \( u = -m^{-1} \sum_{i=0}^{m-1} g(i) \) and \( f(x) = u + g(x) \). It follows that \( \sum_{i=0}^{m-1} f(i) = 0 \), which gives the reproducibility of \( f \) due to the Corollary 18.
Remark 2. Generally, it does not exist a reproducible sequence $f$ such that $f(\mathbb{Z}) = M$. For example, take $M = \{3, 9\} \subset \{0, 3, 6, 9\} \subset \mathbb{Z}_{12}$. By supposing that a reproducible $f$ may exist such that $f(\mathbb{Z}) = M$, let $m = 2^r d$ be the period of $f$, where $d$ is odd. By Theorem 14 one has $\sum_{i=0}^{d-1} f(2^r i) = 0$. Since $f(2^r i) \in \{3, 9\}$, it follows that some integers $k_1, k_2$ must exist such that $3k_1 + 9k_2 = 0 \mod 12$, $k_1 + k_2 = d$. This means that
\[
3k_1 + 9(d - k_1) = 0 \mod 12, \\
6k_1 + 9d = 0 \mod 12, \\
2 \cdot 9d = -2 \cdot 6k_1 = 0 \mod 12,
\]
\[3d = 0 \mod 2.
\]
But because of the oddness of $d$ one would conclude that $3 = 0 \mod 2$, which is absurd.

Example 6. Modal classes of sets of values of $\mathbb{Z}_{12}$-valued reducible sequences

\[
\begin{align*}
\{0\} & \quad + \quad \{0\} \quad = \quad \{0\}, \\
\{0\} & \quad + \quad \{6, 6\} \quad = \quad \{6, 6\}, \\
\{0\} & \quad + \quad \{9, 3\} \quad = \quad \{9, 3\}, \\
\{0\} & \quad + \quad \{6, 3, 3\} \quad = \quad \{6, 3, 3\}, \\
\{0\} & \quad + \quad \{3, 3, 3, 3\} \quad = \quad \{3, 3, 3, 3\}, \\
\{8, 4\} & \quad + \quad \{0\} \quad = \quad \{8, 4\}, \\
\{8, 4\} & \quad + \quad \{6, 6\} \quad = \quad \{4, 2, 4, 2\}, \\
\{8, 4\} & \quad + \quad \{9, 3\} \quad = \quad \{5, 3, 1, 3\}, \\
\{8, 4\} & \quad + \quad \{6, 3, 3\} \quad = \quad \{1, 2, 1, 3, 2, 3\}, \\
\{8, 4\} & \quad + \quad \{3, 3, 3, 3\} \quad = \quad \{1, 2, 1, 2, 1, 2, 1, 2\}, \\
\{4, 4, 4\} & \quad + \quad \{0\} \quad = \quad \{4, 4, 4\}, \\
\{4, 4, 4\} & \quad + \quad \{6, 6\} \quad = \quad \{2, 2, 2, 2, 2\}, \\
\{4, 4, 4\} & \quad + \quad \{9, 3\} \quad = \quad \{3, 1, 3, 1, 3, 1, 3, 1\}, \\
\{4, 4, 4\} & \quad + \quad \{6, 3, 3\} \quad = \quad \{1, 1, 2, 1, 2, 1, 1, 2\}, \\
\{4, 4, 4\} & \quad + \quad \{3, 3, 3, 3\} \quad = \quad \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}.
\end{align*}
\]

8. The reproducibility of rarefied and repeating sequences

Let $f$ be a periodic $\mathbb{Z}_n$-valued sequence and let $d$ be a positive integer.

Definition 6. The rarefied sequence $f_{\text{rat}}$ obtained from the sequence $f$ is given by the following relation:
The integer $d$ is called the factor of insertion.

**Definition 7.** The repeating sequence obtained from $f$ is the sequence $f_{rpt}$ defined by

$$f_{rpt}(x) = f \left( \left\lfloor \frac{x}{d} \right\rfloor \right)$$

where $\lfloor r \rfloor$ means the highest integer $\leq r$. The integer $d$ is called the factor of repetition.

We will write $f_{\text{rar},d}$ and $f_{\text{rpt},d}$ when we want to make evident the presence of the factor $d$. It follows that

$$f_{\text{rar},d_1,d_2} = (f_{\text{rar},d_1})_{\text{rar},d_2},$$

$$f_{\text{rpt},d_1,d_2} = (f_{\text{rpt},d_1})_{\text{rpt},d_2}. \tag{1}$$

**Problem.** Find the relationship between the reproducibility of $f$, $f_{\text{rar}}$ and $f_{\text{rpt}}$.

**Lemma 22.**

$$f_{\text{rpt}} = \sum_{i=0}^{d-1} T^i f_{\text{rar}}.$$

**Proof.** This result is a trivial consequence of the definition of rarefied and repeating sequences. \hfill \Box

**Definition 8.** A sequence $f$ is called $d$-reproducible if $(T^d - 1)^m f = 0$ for an integer $m \geq 1$.

**Lemma 23.** If $f$ is a $d$-reproducible sequence then $f$ is reproducible.

**Proof.** Let $f$ be $d$-reproducible. We write down the decomposition

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep}.$$

It follows that $(T^d - 1)^m f = f$ and $(T - 1)^m f_{\text{red}} = D^m f_{\text{red}} = 0$. According to the identity

$$T^d - 1 = P(T)(T - 1), \quad P(T) = \sum_{i=0}^{d-1} T^i, \tag{2}$$

one has that

$$f = (T^d - 1)^m f = P(T)^m (T - 1)^m f = P(T)^m (T - 1)^m f_{\text{red}} + P(T)^m (T - 1)^m f_{\text{rep}} = P(T)^m (T - 1)^m f_{\text{rep}} \in \text{Rep}. \hfill \Box$$
Theorem 24. The reproducibility of $f$ is equivalent to that of $f_{\text{rar}}$.

Proof. If $f$ is reproducible, the relation

$$(T^d - 1)f_{\text{rar}} = ((T - 1)f)_{\text{rar}}$$

shows that $f_{\text{rar}}$ is $d$-reproducible, i.e., reproducible by the Lemma 23.

Now let suppose that $f_{\text{rar}}$ is reproducible. Then $f = \sum_j f_j$ is the decomposition of $f$ corresponding to the decomposition of $\mathbb{Z}_n$ in $p$-groups. Since the decomposition of $f_{\text{rar}}$ is $\sum_j (f_j)_{\text{rar}}$, it is enough to consider the case where $n = p^k$, $p$ prime. According to (1), we do not loose generality by taking $d$ which is prime. By writing down the decomposition

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, f_{\text{rep}} \in \text{Rep},$$

$$f_{\text{rar}} = (f_{\text{red}})_{\text{rar}} + (f_{\text{rep}})_{\text{rar}},$$

one has that $f_{\text{rar}} \in \text{Rep}$ and $(f_{\text{rep}})_{\text{rar}} \in \text{Rep}$ because of i), i.e., $(f_{\text{red}})_{\text{rar}} \in \text{Rep}$ and one is led to show that if $f \in \text{Red}(\mathbb{Z}_{p^k})$ and $f_{\text{rar}} \in \text{Rep}$, then $f = 0$. Now, since $f$ is reducible, it has to be $p^m$-periodic. Therefore $f_{\text{rar}}$ is $dp^m$-periodic. If $d = p$ then $f_{\text{rar}}$ is reducible too, i.e., $f_{\text{rar}} \in \text{Rep} \cap \text{Red} = \{0\}$ which means that $f = 0$. If $d \neq p$, the reproducibility of $f_{\text{rar}}$ is expressed by the relations

$$\sum_{i=0}^{d-1} f_{\text{rar}} (ip^m + j) = 0, \quad 0 \leq j < p^m. \quad (3)$$

The only terms different from zero in the left side of (3) are those for which

$$ip^m + j = 0 \mod d. \quad (4)$$

Since $d \wedge p = 1$, the equation (4) in the variable $i$ has an unique solution $i(j)$ for all $j$. It follows that in the left part of (3), just one term is zero, i.e.,

$$f_{\text{rar}}(i(j)p^m + j) = f\left(\frac{i(j)p^m + j}{d}\right).$$

This means that

$$f\left(\frac{i(j)p^m + j}{d}\right) = 0, \quad 0 \leq j < p^m. \quad (5)$$

The numbers $(i(j)p^m + j)/d, 0 \leq j < p^m$, form a fundamental system of classes mod $p^m$. In fact, if

$$\frac{i(j_1)p^m + j_1}{d} = \frac{i(j_2)p^m + j_2}{d} \mod p^m,$$

then

$$i(j_1)p^m + j_1 = i(j_2)p^m + j_2 \mod p^m,$$

i.e., $j_1 = j_2 \mod p^m$ which means that $j_1 = j_2$. According to (5) and to the $p^m$-periodicity of $f$, one obtains $f = 0$. \qed
Lemma 25. If \( d \wedge n = 1 \) and if \( \sum_{i=0}^{d-1} T^i f \) is reproducible then \( f \) is reproducible.

Proof. Let's write down the decomposition

\[ f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \ f_{\text{rep}} \in \text{Rep}, \]

where \( P(T) \) is given by (2). Since \( P(T)f \in \text{Rep} \) (by hypothesis) and also \( P(T)f_{\text{rep}} \in \text{Rep} \), one has \( P(T)f_{\text{red}} \in \text{Red} \cap \text{Rep} = \{0\} \). This leads to the relation

\[ (T^d - 1)f_{\text{red}} = (T - 1)P(T)f_{\text{red}} = 0, \]

which means that \( f_{\text{red}} \) is \( d \)-periodic. Since \( f_{\text{red}} \) is reducible, it is also \( m \)-periodic where \( m \) is an integer that divides \( n \). This means that \( f \) is \( d \wedge m \)-periodic, which gives the 1-periodicity, i.e., \( f \) is a constant. In conclusion

\[ f = z + f_{\text{rep}}, \quad z \in \mathbb{Z}_n, \ f_{\text{rep}} \in \text{Rep}, \]

\[ P(T)f = dz + P(T)f_{\text{rep}}. \]

By using the same argument \( dz \in \text{Red} \cap \text{Rep} = \{0\} \), i.e., \( z = 0 \) since the multiplication by \( d \) is an automorphism of \( \mathbb{Z}_n \).

Theorem 26. The reproducibility of \( f \) is equivalent to that of \( f_{\text{rpt}} \).

Proof. If \( f \) is reproducible, then (Theorem 24) \( f_{\text{rpt}} \) is reproducible, which means that (Lemma 22) \( f_{\text{rpt}} \) is reproducible, too.

Let's now suppose that \( f_{\text{rpt}} \) is reproducible. As in the proof of Theorem 24 one may just consider the case where \( f \) takes its values in a \( p \)-group and \( d \) is prime. If \( d = p \) one uses the same argument as in the proof of Theorem 24. If \( d \neq p \), then (Lemmas 22 and 25) \( f_{\text{rpt}} \) is reproducible, which leads to the conclusion that (Theorem 24) \( f \) is reproducible too.

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Moreno Andreatta
Equipe Représentations Musicales
Ircam/CNRS — Centre G. Pompidou
Paris
FRANCE
E-mail: Moreno.Andreatta@ircam.fr

Dan T. Vuză
Department of Mathematics
University of Bucharest
Bucharest
ROMANIA
E-mail: Dan.Vuză@imar.ro