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ON SOME PROPERTIES OF PERIODIC SEQUENCES IN ANATOL VIERU'S MODAL THEORY

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ABSTRACT. Algebraic methods have been currently applied to music in the second half of the twentieth-century (see [M. Andreatta: *Group-theoretical Methods applied to Music*, unpublished dissertation, 1997], [M. Chemilier: *Structure et Méthode algébraiques en informatique Musicale.* Thèse de doctorat, L. I. T. P., Institut Blaise Pascal, 1990] and [G. Mazzola et al.: *The Topos of Music—Geometric Logic of Concepts, Theory and Performance*] for main references). By starting from Anatol Vieru's compositional technique based on finite difference calculus on periodic modal sequences, as it has been introduced in his book [*Cartea modurilor, 1* (*Le livre des modes, 1*). Ed. Muzicala, Bucarest, 1980. Revised ed. *The book of modes*, 1993], the present essay tries to generalize some properties by means of abstract group theory. Two main classes of periodic sequences are considered: reducible and reproducible sequences, replacing respectively Vieru's modal and irreducible sequences. It turns out that any periodic sequence can be decomposed in a unique way into a reducible and a reproducible component.

1. Reducible and reproducible sequences

For any sequence f defined on $\mathbb{Z}$ taking values in a finite abelian group G we define the translated sequence Tf and the sequence of differences Df by

$$Tf(x) = f(x+1), \quad Df(x) = f(x+1) - f(x).$$

The relationship between the translated sequence and the sequence of differences is expressed by the following equation:

$$D = T - 1.$$

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Definition 1. The sequence f is called *m-periodic* if $f(x+m) = f(x)$ for any $x \in \mathbb{Z}$.

Affirming that f is m-periodic, it is equivalent to the relation $T^m f = f$. If f is m-periodic, then Tf and Df are also m-periodic.

Definition 2. The sequence f is called *reducible* if an integer $k \geq 0$ does exist such that $D^k f = 0$.

The sequence f is called *reproducible* if an integer $k \geq 0$ does exist such that $D^k f = f$.

By Red(G) and Rep(G) we will designate respectively the set of reducible and reproducible sequences taking values in G (also called G-valued sequences).

Example 1. Example of a reducible sequence. In Anatol V i e r u's book [10] a $\mathbb{Z}_{12}$-valued sequence of period 72 is considered. It can be represented by means of 6 lines of 12 elements

$$
\begin{array}{cccccccccccc}
(4 & 1 & 0 & 8 & 8 & 7 & 0 & 6 & 8 & 1 & 4 & 0 \\
8 & 11 & 4 & 6 & 0 & 5 & 4 & 4 & 0 & 11 & 8 & 10 \\
0 & 9 & 8 & 4 & 4 & 3 & 8 & 2 & 4 & 9 & 0 & 8 \\
4 & 7 & 0 & 2 & 8 & 1 & 0 & 0 & 8 & 7 & 4 & 6 \\
8 & 5 & 4 & 0 & 0 & 11 & 4 & 10 & 0 & 5 & 8 & 4 \\
0 & 3 & 8 & 10 & 4 & 9 & 8 & 8 & 4 & 3 & 0 & 2).
\end{array}
$$

Any column is a periodic sequence obtained by adding modulo 12 a constant value to a basis element (i.e., the top of the column). By putting the constant value as an index for the basis element, one has the following compact expression

$$
\begin{aligned}
&(4_4 \;\; 1_{10} \;\; 0_4 \;\; 8_{10} \;\; 8_4 \;\; 7_{10} \;\; 0_4 \;\; 6_{10} \;\; 8_4 \;\; 1_{10} \;\; 4_4 \;\; 0_{10}), \\
&D^1 = (2_6\; 9_6\; 11_6\; 8_6\; 0_6\; 11_6\; 5_6\; 6_6\; 2_6\; 5_6\; 3_6\; 8_6), \\
&D^2 = (0\; 7\; 2\; 9\; 4\; 11\; 6\; 1\; 8\; 3\; 10\; 5), \\
&D^3 = (7), \\
&D^4 = 0.
\end{aligned}
$$

Example 2. Example of a reproducible sequence

$$
\begin{aligned}
f &= (8\; 11\; 0\; 1\; 4\; 0), \\
D^1 &= (3\; 1\; 1\; 3\; 8\; 8), \\
D^2 &= (10\; 0\; 2\; 5\; 0\; 7), \\
D^3 &= (2\; 2\; 3\; 7\; 7\; 3), \\
D^4 &= f.
\end{aligned}
$$

Theorem 1. *Let d_i, $0 \le i \le N$, be some integers such that at least one of them is relatively prime to the number of elements of G. Then, an integer m does exist such that any sequence verifying*

$$\sum_{i=0}^{N} d_i T^i f = 0$$

is m-periodic.

Corollary 2. *Reducibles and reproducibles sequences are periodic.*

Theorem 3. *All periodic sequence can be decomposed in a unique way*

$$f = f_{\text{red}} + f_{\text{rep}}\,, \qquad f_{\text{red}} \in \text{Red(G)}\,, \;\; f_{\text{rep}} \in \text{Rep(G)}\,.$$

P r o o f. Let f be m-periodic. Being the collection of sequences m-periodic finite, two integers $k, l \ge 1$ do exist such that $D^k f = D^{k+l} f$. By induction on r one has $D^k f = D^{k+rl} f$. In the same way it can be shown that two integers $r, s \ge 1$ may exist such that $D^{rl} f = D^{(r+s)l} f$. We put

$$f_{\text{red}} = f - D^{rl} f, \qquad f_{\text{rep}} = D^{rl} f\,.$$

It follows that $D^k(f - D^{rl} f) = 0$, $D^{sl} D^{rl} f = D^{rl} f$, which means that f_{red} and f_{rep} give the needed decomposition.

The unicity comes from the relation $\text{Red(G)} \cap \text{Rep(G)} = \{0\}$. □

2. Decomposition of $\mathbb{Z}_n$ into p-groups

There is a one-to-one relation between the subgroups of $\mathbb{Z}_n$ and the family of integers d that divide n by $1 \le d \le n$, i.e., for any such d we may take the unique subgroup of $\mathbb{Z}_n$ with d elements. The latter can be characterized as the set of $z \in \mathbb{Z}_n$ such that $dz = 0$ or, equivalently, as the set $\frac{n}{d}\mathbb{Z}_n$ of elements having the form $\frac{n}{d}z$ where z belongs to $\mathbb{Z}_n$.

Definition 3. The abelian group G is a *direct sum of a family of subgroups* $G_1, \ldots, G_m$ of G if any $x \in G$ may be decomposed in a unique way into a sum $x_1 + \cdots + x_m$ with $x_i \in G_i$ for $1 \le i \le m$.

We will put $G = \bigoplus_{i=1}^{m} G_i$.

Definition 4. Let p be a prime number. A finite abelian group is called *p-group* if its cardinality is a power of p.

THEOREM 4. *Any group $\mathbb{Z}_n$ is a direct sum of its p-maximals subgroups.*

If $n = \prod_{i=1}^{m} p^{k_i}$ is the decomposition of n into prime factors, the decomposition of $\mathbb{Z}_n$ in maximals p-subgroups can be written as follows

$$\mathbb{Z}_n = \bigoplus_{i=1}^{m} G_{p^{k_i}},$$

where $G_{p^{k_i}}$ is the subgroup of $\mathbb{Z}_n$ with p^{k_i} elements. The decomposition of any $z \in \mathbb{Z}_n$ defines the elements $\pi_i(z) \in G_{p^{k_i}}$ in a unique way such that $z = \sum_{i=1}^{m} \pi_i(z)$. The arrows $\pi_i \colon \mathbb{Z}_n \to G_{p^{k_i}}$ are group morphisms.

The p_i-component $\pi_i(z)$ of z is the unique element $y \in \mathbb{Z}_n$ satisfying the relations $p^{k_i} y = \frac{n}{p^{k_i}}(z - y) = 0$ in $\mathbb{Z}_n$. Let q_i be an integer verifying

$$q_i \frac{n}{p^{k_i}} = 1 \bmod p^{k_i}\,.$$

It follows that $p^{k_i} \frac{n}{p^{k_i}} q_i z = \frac{n}{p^{k_i}}(z - \frac{n}{p^{k_i}} q_i z) = 0$ in $\mathbb{Z}_n$, which means that

$$\pi_i(z) = \frac{n}{p^{k_i}} q_i z\,.$$

EXAMPLE 3. Subgroups of $\mathbb{Z}_{12}$

G_1	$\{0\}$,	
G_2	$\{0,\ 6\}$,	
G_3	$\{0,\ 4,\ 8\}$,	maximal 3-group,
G_4	$\{0,\ 3,\ 6,\ 9\}$,	maximal 2-group,
G_6	$\{0,\ 2,\ 4,\ 6,\ 8,\ 10\}$,	
G_{12}	$\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11\}$.	

EXAMPLE 4. Decomposition of $\mathbb{Z}_{12}$ into p-groups

$$\mathbb{Z}_{12} = G_3 \bigoplus G_4\,.$$

	0	3	6	9
0	0	3	6	9
4	4	7	10	1
8	8	11	2	5

$$q_3 = 1\,, \quad 1 \cdot \frac{12}{3} = 1 \bmod 3\,,$$
$$q_4 = -1\,, \quad -1 \cdot \frac{12}{4} = 1 \bmod 4\,,$$
$$\pi_3(x) = \frac{12}{3}\, q_3 x = 4x\,, \quad \pi_4(x) = \frac{12}{4}\, q_4 x = -3x\,.$$

$$5 = 8 + 9 = 4 \cdot 5 - 3 \cdot 5 = 4 \cdot 2 - 3 \cdot 1\,,$$
$$7 = 4 + 3 = 4 \cdot 7 - 3 \cdot 7 = 4 \cdot 1 - 3 \cdot 3\,,$$
$$11 = 8 + 3 = 4 \cdot 11 - 3 \cdot 11 = 4 \cdot 2 - 3 \cdot 3\,.$$

Remark 1. Many theoretical works ([2], [8] and [10]) have already shown the musical interest of the decomposition of the cyclic group $\mathbb{Z}_{12}$ into a direct sum of its maximal p-groups from a theoretical, analytical and compositional point of view.

Theorem 5. *Let $\varphi_i \colon \mathbb{Z}_n \to \mathbb{Z}_{p^{k_i}}$ be the canonic map. The map $\varphi \colon \mathbb{Z}_n \to \prod_{i=1}^{m} \mathbb{Z}_{p^{k_i}}$ defined by*

$$\varphi(z) = \bigl(\varphi_1(z), \dots, \varphi_m(z)\bigr)$$

is a ring homomorphism with inverse given by

$$(z_1, \dots, z_m) \mapsto \sum_{i=1}^{m} \frac{n}{p^{k_i}}\, q_i z_i\,.$$

For any $\mathbb{Z}_n$-valued sequence f we put $f_i(x) = \pi_i\bigl(f(x)\bigr)$. We will say that $f = \sum_i f_i$ is the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups.

3. Characterization of reducible sequences

Proposition 6. *Let $f = \sum_j f_j$ be the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups. Then $f \in \mathrm{Red}(\mathbb{Z}_n)$ if and only if $f_j \in \mathrm{Red}(\mathbb{Z}_n)$ for any j.*

Theorem 7. *Let f be a $\mathbb{Z}_{p^k}$-valued sequence. Then f is reducible if and only if it is p^m-periodic for a given $m \geq 0$.*

P r o o f. By induction on k. For $k = 1$ the result comes from the following lemma. □

Lemma 8. *For any $\mathbb{Z}_p$-valued sequence f, one has*

$$(T-1)^{p^m} f = \left(T^{p^m} - 1\right) f\,.$$

Proof. By induction on m one is conducted to the case $m=1$.

$$(T-1)^p f = \sum_{i=0}^{p} (-1)^{p-i} C_p^i \, T^i f\,.$$

Since p divides any C_p^i by $1 \le i \le p-1$ and since $pf = 0$ it follows that the sum of the right member is reduced to $T^p f - f$.

Let now the theorem be true for any $k_1 < k$. We designate f_1 the sequence defined by $f_1(x) = pf(x)$. It is clear that f_1 takes values in the subgroup $G_{p^{k-1}}$ with p^{k-1} elements of $\mathbb{Z}_{p^k}$, isomorphic to $\mathbb{Z}_{p^{k-1}}$.

Let f be p^m-periodic. Then f_1 is also p^m-periodic, i.e., reducible (induction hypothesis). Therefore, there exists $l \ge 1$ such that $D^l f_1 = 0$. By definition of $f_{1,}$ it means that $p(D^l f)(x) = 0$ for any x, i.e., the sequence $D^l f$ takes values into the subgroup G_p with p elements of $\mathbb{Z}_{p^k}$, isomorphic to $\mathbb{Z}_p$. Since $D^l f$ is p^m-periodic, the induction hypothesis leads to the existence of $l_1 \ge 1$ such that $D^{l_1} D^l f = 0$, i.e., $D^{l_1+l} f = 0$ and f is reducible.

Let f be reducible. Then $f_1 \in \mathrm{Red}(G_{p^{k-1}})$ by induction hypothesis it follows that there exists $m_1 \ge 0$ such that f_1 is p^{m_1}-periodic. This means that $f(x) - f(x+p^{m_1}) \in G_p$ for any x. We define the sequence f_2 by $f_2(x) = f(x) - f(x + p^{m_1})$. Since $f_2 \in \mathrm{Red}(G_p)$, the induction hypothesis guaranties the existence of $m_2 \ge 0$ such that f_2 is p^{m_2}-periodic. In short, $(1 - T^{p_{m_2}})(1 - T^{p_{m_1}}) f = 0$. To conclude we just need the following. □

Lemma 9. *Let f be a $\mathbb{Z}_n$-valued sequence such that $\left(1-T^k\right)\left(1-T^l\right) f = 0$. Then f is kln-periodic.*

Proof. From $\left(1-T^l\right) f = T^k \left(1-T^l\right) f$ one deduces that $\left(1-T^l\right) f = T^{ki}\left(1-T^l\right) f$ for any $i \ge 1$, and in particular that $\left(1-T^{kl}\right)\left(1-T^l\right) f = 0$. By the same argument, $\left(1-T^{kl}\right) f = T^{li}\left(1-T^{kl}\right) f$ for any $i \ge 1$. This means that

$$\left(1-T^{kln}\right) f = \left(1-\left(T^{kl}\right)^n\right) f = \left(\sum_{i=0}^{n-1} T^{kli}\right)\left(1-T^{kl}\right) f = n\left(1-T^{kl}\right) f = 0\,.$$

□

Corollary 10. $\mathrm{Red}(\mathbb{Z}_n)$ *is a ring.*

Corollary 11. *Let $f \in \mathrm{Red}(\mathbb{Z}_n)$ be such that $f(x)$ is invertible in $\mathbb{Z}_n$ for all x. Then $f^{-1} \in \mathrm{Red}(\mathbb{Z}_n)$.*

Corollary 12. *Let k, l be two integers and let g be defined by $g(x) = f(kx+l)$. If $f \in \mathrm{Red}(\mathbb{Z}_n)$ then $g \in \mathrm{Red}(\mathbb{Z}_n)$.*

4. Characterization of reproducible sequences

Proposition 13. *Let $f = \sum_j f_j$ be the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups. Then $f \in \mathrm{Rep}(\mathbb{Z}_n)$ if and only if $f_j \in \mathrm{Rep}(\mathbb{Z}_n)$ for any j.*

Definition 5. For every m-periodic sequence f and for every integer d dividing m, we define the *d-periodised* obtained from f as the sequence $\sum_{i=0}^{m/d-1} T^{id} f$.

The equation

$$(1 - T^d) \sum_{i=0}^{m/d-1} (T^d)^i f = (1 - (T^d)^{m/d}) f = (1 - T^m) f = 0$$

shows that the d-periodised sequence is indeed a d-periodic sequence.

Theorem 14. *Let f be a $\mathbb{Z}_{p^k}$-valued m-periodic sequence. Then f is reproducible if and only if the p^r-periodised of f is zero, where p^r is the highest power of p that divides m.*

Proof. If f is reproducible, then its p^r-periodised is also reproducible by construction, but also reducible according to Theorem 7, because it is p^r-periodic. Then the p^r-periodised is zero, because $\mathrm{Rep} \cap \mathrm{Red} = \{0\}$.

Conversely, by supposing that the p^r-periodised is zero, we may write the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}, \qquad f_{\mathrm{red}} \in \mathrm{Red}, \; f_{\mathrm{rep}} \in \mathrm{Rep}.$$

According as we have shown, the p^r-periodised of f_{rep} is zero. Then the previous equation shows that the p^r-periodised of f_{red} must be zero too. Due to Theorem 7, f_{red} must be p^s-periodised for a given integer $s \geq 0$. Because of the fact that the m-periodised of f implies that of f_{red}, the period of the latter divides m and p^s, which shows that f_{red} is in fact p^r-periodic. As a consequence its p^r-periodised is equal to $\frac{m}{p^r} f_{\mathrm{red}}$. Since $\frac{m}{p^r} \wedge p^k = 1$, the multiplication by $\frac{m}{p^r}$ is an automorphism of $\mathbb{Z}_{p^k}$. From $\frac{m}{p^r} f_{\mathrm{red}} = 0$ one may conclude that $f_{\mathrm{red}} = 0$.

□

Corollary 15. *Let f be a $\mathbb{Z}_{p^k}$-valued m-periodic sequence and let p^r be the highest power of p dividing m. Then f is reproducible if and only if the relations*

$$\sum_{i=0}^{m/p^r-1} f(p^r i + x) = 0$$

hold for all x such that $0 \leq x < p^r$.

5. Calculation of reducible and reproducible components of a periodic sequence

Proposition 16. *Let $f = \sum_j f_j$ be the decomposition of the periodic sequence f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups and let $f_j = f_{j,\mathrm{red}} + f_{j,\mathrm{rep}}$ be the decomposition of f_j in a sum of a reducible and a reproducible sequence. Then the decomposition of f*

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$

is given by

$$f_{\mathrm{red}} = \sum_j f_{j,\mathrm{red}}\,,\ f_{\mathrm{rep}} = \sum_j f_{j,\mathrm{rep}}\,.$$

Theorem 17. *Let f be a $\mathbb{Z}_{p^k}$-valued m-periodic sequence and let f_{per} be the p^r-periodised of f, where p^r is the highest power of p dividing m. Let $\left(\frac{m}{p^r}\right)^{-1}$ be the inverse of $\frac{m}{p^r}$ $\bmod p^k$. Then the decomposition of f*

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$

is given by

$$f_{\mathrm{red}} = \left(\frac{m}{p^r}\right)^{-1} f_{\mathrm{per}}\,, \qquad f_{\mathrm{rep}} = f - \left(\frac{m}{p^r}\right)^{-1} f_{\mathrm{per}}\,.$$

Proof. Since f_{red} is p^r-periodic by construction, it is reducible according to the Theorem 7. The p^r-periodised of f_{rep} is zero:

$$\sum_{i=0}^{m/p^r-1} T^{ip^r} f_{\mathrm{rep}} = \sum_{i=0}^{m/p^r-1} T^{ip^r} f - \left(\frac{m}{p^r}\right)\left(\frac{m}{p^r}\right)^{-1} f_{\mathrm{per}} = f_{\mathrm{per}} - f_{\mathrm{per}} = 0\,.$$

Due to the Theorem 14, f_{rep} is reproducible. □

Corollary 18. *Let m and n be two integers such that $m \wedge n = 1$ and let f be a $\mathbb{Z}_n$-valued m-periodic sequence. f is reproducible if and only if $\sum_{i=0}^{m-1} f(i) = 0$.*

6. Decomposition algorithm

1. Write the decomposition of n

$$n = \prod_{j=1}^{N} p_j^{k_j}$$

into prime factors.

2. Find the integers q_j such that
$$q_j \frac{n}{p_j^{k_j}} = 1 \mod p_j^{k_j}.$$
3. For all j build the sequences $f_{j,\mathrm{red}}$ and $f_{j,rep}$ as follows.
4. Set $f_j(x) = \varphi_j\big(f(x)\big)$, where $\varphi_j : \mathbb{Z}_n \to \mathbb{Z}_{p_j^{k_j}}$ is the canonic map.
5. Let $p_j^{r_j}$ be the highest power of p_j dividing the period m of f. Determine the inverse $\left(\frac{m}{p_j^{r_j}}\right)^{-1}$ of $\frac{m}{p_j^{r_j}}$ mod $p_j^{k_j}$.
6. Write down the m elements of the period of f_j as a table with $\frac{m}{p_j^{r_j}}$ lines and $p_j^{r_j}$ columns (if $r_j = 0$, the table will have an unique column). Add a line given by the elements of any column and by multiplying by $\left(\frac{m}{p_j^{r_j}}\right)^{-1}$ module $p_j^{k_j}$.
7. Let $f_{j,\mathrm{red}}$ be the $p_j^{r_j}$-periodic sequence defined by the line built at the previous step and build $f_{j,\mathrm{rep}} = f_j - f_{j,\mathrm{red}}$.
8. By setting
$$f_{\mathrm{red}} = \sum_{i=1}^{N} q_j \frac{n}{p_j^{k_j}} f_{j,\mathrm{red}}\,, \qquad f_{\mathrm{rep}} = \sum_{i=1}^{N} q_j \frac{n}{p_j^{k_j}} f_{j,\mathrm{rep}}\,.$$
we have the decomposition of any periodic sequence in a reducible and reproducible component.

Example 5. Decomposition in reducible and reproducible components. Let f be the following $\mathbb{Z}_{12}$-valued sequence.
$$f = (0\ 0\ 7\ 7\ 4\ 4\ 3\ 3\ 4\ 4\ 7\ 7).$$
The sequences corresponding to the decomposition of $\mathbb{Z}_{12} = G_3 \bigoplus G_4$ are respectively
$$f_1 = (0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1)$$
$$f_2 = (0\ 0\ 3\ 3\ 0\ 0\ 3\ 3\ 0\ 0\ 3\ 3)$$

	0	0	1
	1	1	1
	0	0	1
	1	1	1
	2	2	1
$\div 4$ mod 3	2	2	1
$\times 4$ mod 12	8	8	4

	0	0	3	3
	0	0	3	3
	0	0	3	3
	0	0	1	1
$\div 3 \bmod 4$	0	0	3	3
$\times -3 \bmod 12$	0	0	3	3

	8	8	4	8	8	4	8	8	4	8	8	4
	0	0	3	3	0	0	3	3	0	0	3	3
f_{red}	8	8	7	11	8	4	11	11	4	8	11	7
f_{rep}	4	4	0	8	8	0	4	4	0	8	8	0

7. The sets of values of reducible and reproducible sequences

THEOREM 19. *Let $\mathbb{Z}_n = \bigoplus_j G_j$ be the decomposition of $\mathbb{Z}_n$ in p-groups. A subset M of $\mathbb{Z}_n$ is the set of values of a reducible sequence if and only if two subsets $M_j \subset G_j$ do exist such that $M = \sum_j M_j$.*

This theorem is a direct consequence of the the following proposition.

PROPOSITION 20. *Let $\mathbb{Z}_n = \bigoplus_j G_j$ be the decomposition of $\mathbb{Z}_n$ in p-groups and let $f = \sum_j f_j$ be the decomposition of the reducible sequence f corresponding to the decomposition of $\mathbb{Z}_n$. Then $f(\mathbb{Z}) = \sum_j f_j(\mathbb{Z})$.*

Proof. Clearly $f(\mathbb{Z}) \subset \sum_j f_j(\mathbb{Z})$. On the other hand, let $y_j = f_j(x_j) \in f_j(\mathbb{Z})$ and, for all integer p_j dividing n, let p_j^r be the highest power of p_j dividing the period m of f. According to the Theorem 7, f_j is p_j^r-periodic. A well known result of algebra gives the existence of $x \in \mathbb{Z}$ verifying $x = x_j \bmod p_j^r$ for all j. It follows that $f_j(x) = f_j(x_j)$ for all j, i.e.,

$$\sum_j y_j = \sum_j f_j(x_j) = \sum_j f_j(x) = f(x) \in f(\mathbb{Z}).$$

□

THEOREM 21. *For any subset $M \subset \mathbb{Z}_n$ it does exist an element $u \in \mathbb{Z}_n$ and a reproducible sequence f such that $f(\mathbb{Z}) = u + M$.*

Proof. It does exist an integer m such that $m \wedge n = 1$ and the m-periodic sequence g such that $g(\mathbb{Z}) = M$. Since $m \wedge n = 1$, the inverse m^{-1} of m mod n does exist. We set $u = -m^{-1} \sum_{i=0}^{m-1} g(i)$ and $f(x) = u + g(x)$. It follows that $\sum_{i=0}^{m-1} f(i) = 0$, which gives the reproducibility of f due to the Corollary 18.

□

Remark 2. Generally, it does not exist a reproducible sequence f such that $f(\mathbb{Z}) = M$. For example, take $M = \{3,9\} \subset \{0,3,6,9\} \subset \mathbb{Z}_{12}$. By supposing that a reproducible f may exist such that $f(\mathbb{Z}) = M$, let $m = 2^r d$ be the period of f, where d is odd. By Theorem 14 one has $\sum_{i=0}^{d-1} f(2^r i) = 0$. Since $f(2^r i) \in \{3,9\}$, it follows that some integers k_1, k_2 must exist such that $3k_1 + 9k_2 = 0 \mod 12$, $k_1 + k_2 = d$. This means that

$$\begin{aligned} 3k_1 + 9(d - k_1) &= 0 \mod 12\,, \\ 6k_1 + 9d &= 0 \mod 12\,, \\ 2 \cdot 9d = -2 \cdot 6k_1 &= 0 \mod 12\,, \\ 3d &= 0 \mod 2\,. \end{aligned}$$

But because of the oddness of d one would conclude that $3 = 0 \mod 2$, which is absurd.

Example 6. Modal classes of sets of values of $\mathbb{Z}_{12}$-valued reducible sequences

$$\begin{array}{llllll}
\{0\} & + & \{0\} & = & \{0\}\,, \\
\{0\} & + & \{6,6\} & = & \{6,6\}\,, \\
\{0\} & + & \{9,3\} & = & \{9,3\}\,, \\
\{0\} & + & \{6,3,3\} & = & \{6,3,3\}\,, \\
\{0\} & + & \{3,3,3,3\} & = & \{3,3,3,3\}\,, \\
\{8,4\} & + & \{0\} & = & \{8,4\}\,, \\
\{8,4\} & + & \{6,6\} & = & \{4,2,4,2\}\,, \\
\{8,4\} & + & \{9,3\} & = & \{5,3,1,3\}\,, \\
\{8,4\} & + & \{6,3,3\} & = & \{1,2,1,3,2,3\}\,, \\
\{8,4\} & + & \{3,3,3,3\} & = & \{1,2,1,2,1,2,1,2\}\,, \\
\{4,4,4\} & + & \{0\} & = & \{4,4,4\}\,, \\
\{4,4,4\} & + & \{6,6\} & = & \{2,2,2,2,2,2\}\,, \\
\{4,4,4\} & + & \{9,3\} & = & \{3,1,3,1,3,1,3,1\}\,, \\
\{4,4,4\} & + & \{6,3,3\} & = & \{1,1,2,1,1,2,1,1,2\}\,, \\
\{4,4,4\} & + & \{3,3,3,3\} & = & \{1,1,1,1,1,1,1,1,1,1,1,1\}\,.
\end{array}$$

8. The reproducibility of rarefied and repeating sequences

Let f be a periodic $\mathbb{Z}_n$-valued sequence and let d be a positive integer.

Definition 6. The *rarefied sequence* f_{rar} obtained from the sequence f is given by the following relation:

$$f_{\mathrm{rar}}(x) = \begin{cases} f\left(\frac{x}{d}\right) & \text{if } x = 0 \mod d\,, \\ 0 & \text{if } x \neq 0 \mod d\,. \end{cases}$$

The integer d is called the *factor of insertion.*

DEFINITION 7. The *repeating sequence* obtained from f is the sequence f_{rpt} defined by

$$f_{\mathrm{rpt}}(x) = f\left(\left[\frac{x}{d}\right]\right)$$

where $[r]$ means the highest integer $\leq r$. The integer d is called the *factor of repetition.*

We will write $f_{\mathrm{rar},d}$ and $f_{\mathrm{rpt},d}$ when we want to make evident the presence of the factor d. It follows that

$$f_{\mathrm{rar},d_1 d_2} = (f_{\mathrm{rar},d_1})_{\mathrm{rar},d_2}\,, \tag{1}$$
$$f_{\mathrm{rpt},d_1 d_2} = (f_{\mathrm{rpt},d_1})_{\mathrm{rpt},d_2}\,.$$

Problem. Find the relationship between the reproducibility of f, f_{rar} and f_{rpt}.

LEMMA 22.

$$f_{\mathrm{rpt}} = \sum_{i=0}^{d-1} T^i f_{\mathrm{rar}}\,.$$

Proof. This result is a trivial consequence of the definition of rarefied and repeating sequences. □

DEFINITION 8. A sequence f is called d-reproducible if $(T^d - 1)^m f = 0$ for an integer $m \geq 1$.

LEMMA 23. *If f is a d-reproducible sequence then f is reproducible.*

Proof. Let f be d-reproducible. We write down the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\ .$$

It follows that $(T^d - 1)^m f = f$ and $(T - 1)^m f_{\mathrm{red}} = D^m f_{\mathrm{red}} = 0$. According to the identity

$$T^d - 1 = P(T)(T-1)\,, \qquad P(T) = \sum_{i=0}^{d-1} T^i, \tag{2}$$

one has that

$$\begin{aligned} f &= (T^d - 1)^m f = P(T)^m (T-1)^m f \\ &= P(T)^m (T-1)^m f_{\mathrm{red}} + P(T)^m (T-1)^m f_{\mathrm{rep}} \\ &= P(T)^m (T-1)^m f_{\mathrm{rep}} \in \mathrm{Rep}\ . \end{aligned}$$

□

THEOREM 24. *The reproducibility of f is equivalent to that of f_{rar}.*

Proof. If f is reproducible, the relation

$$(T^d - 1) f_{\mathrm{rar}} = \big((T-1)f\big)_{\mathrm{rar}}$$

shows that f_{rar} is d-reproducible, i.e., reproducible by the Lemma 23.

Now let suppose that f_{rar} is reproducible. Then $f = \sum_j f_j$ is the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups. Since the decomposition of f_{rar} is $\sum_j (f_j)_{\mathrm{rar}}$, it is enough to consider the case where $n = p^k$, p prime. According to (1), we do not loose generality by taking d which is prime. By writing down the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}, \qquad f_{\mathrm{red}} \in \mathrm{Red}, \; f_{\mathrm{rep}} \in \mathrm{Rep},$$
$$f_{\mathrm{rar}} = (f_{\mathrm{red}})_{\mathrm{rar}} + (f_{\mathrm{rep}})_{\mathrm{rar}}.$$

one has that $f_{\mathrm{rar}} \in \mathrm{Rep}$ and $(f_{\mathrm{rep}})_{\mathrm{rar}} \in \mathrm{Rep}$ because of i), i.e., $(f_{\mathrm{red}})_{\mathrm{rar}} \in \mathrm{Rep}$ and one is led to show that if $f \in \mathrm{Red}(\mathbb{Z}_{p^k})$ and $f_{\mathrm{rar}} \in \mathrm{Rep}$, then $f = 0$. Now, since f is reducible, it has to be p^m-periodic. Therefore f_{rar} is dp^m-periodic. If $d = p$ then f_{rar} is reducible too, i.e., $f_{\mathrm{rar}} \in \mathrm{Rep} \cap \mathrm{Red} = \{0\}$ which means that $f = 0$. If $d \neq p$, the reproducibility of f_{rar} is expressed by the relations

$$\sum_{i=0}^{d-1} f_{\mathrm{rar}}(ip^m + j) = 0, \qquad 0 \le j < p^m. \tag{3}$$

The only terms different from zero in the left side of (3) are those for which

$$ip^m + j = 0 \mod d. \tag{4}$$

Since $d \wedge p = 1$, the equation (4) in the variable i has an unique solution $i(j)$ for all j. It follows that in the left part of (3), just one term is zero, i.e.,

$$f_{\mathrm{rar}}\big(i(j)p^m + j\big) = f\left(\frac{i(j)p^m + j}{d}\right).$$

This means that

$$f\left(\frac{i(j)p^m + j}{d}\right) = 0, \qquad 0 \le j < p^m. \tag{5}$$

The numbers $\big(i(j)p^m + j\big)/d$, $0 \le j < p^m$, form a fundamental system of classes mod p^m. In fact, if

$$\frac{i(j_1)p^m + j_1}{d} = \frac{i(j_2)p^m + j_2}{d} \mod p^m,$$

then

$$i(j_1)p^m + j_1 = i(j_2)p^m + j_2 \mod p^m,$$

i.e., $j_1 = j_2 \mod p^m$ which means that $j_1 = j_2$. According to (5) and to the p^m-periodicity of f, one obtains $f = 0$. □

LEMMA 25. *If $d \wedge n = 1$ and if $\sum_{i=0}^{d-1} T^i f$ is reproducible then f is reproducible.*

Proof. Let's write down the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}, \qquad f_{\mathrm{red}} \in \mathrm{Red}, \; f_{\mathrm{rep}} \in \mathrm{Rep},$$

$$P(T)f = P(T)f_{\mathrm{red}} + P(T)f_{\mathrm{rep}},$$

where $P(T)$ is given by (2). Since $P(T)f \in \mathrm{Rep}$ (by hypothesis) and also $P(T)f_{\mathrm{rep}} \in \mathrm{Rep}$, one has $P(T)f_{\mathrm{red}} \in \mathrm{Red} \cap \mathrm{Rep} = \{0\}$. This leads to the relation

$$(T^d - 1)f_{\mathrm{red}} = (T-1)P(T)f_{\mathrm{red}} = 0,$$

which means that f_{red} is d-periodic. Since f_{red} is reducible, it is also m-periodic where m is an integer that divides n. This means that f is $d \wedge m$-periodic, which gives the 1-periodicity, i.e., f is a constant. In conclusion

$$f = z + f_{\mathrm{rep}}, \qquad z \in \mathbb{Z}_n, \; f_{\mathrm{rep}} \in \mathrm{Rep},$$

$$P(T)f = dz + P(T)f_{\mathrm{rep}}.$$

By using the same argument $dz \in \mathrm{Red} \cap \mathrm{Rep} = \{0\}$, i.e., $z = 0$ since the multiplication by d is an automorphism of $\mathbb{Z}_n$. □

THEOREM 26. *The reproducibility of f is equivalent to that of f_{rpt}.*

Proof. If f is reproducible, then (Theorem 24) f_{rar} is reproducible, which means that (Lemma 22) f_{rpt} is reproducible, too.

Let's now suppose that f_{rpt} is reproducible. As in the proof of Theorem 24 one may just consider the case where f takes its values in a p- group and d is prime. If $d = p$ one uses the same argument as in the proof of Theorem 24. If $d \neq p$, then (Lemmas 22 and 25) f_{rar} is reproducible, which leads to the conclusion that (Theorem 24) f is reproducible too. □

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