Multisymplectic geometry with symmetry. Application to the Reissner beam

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***Abstract*—Although acoustics is one of the disciplines of mechanics, its "geometrization" is still limited to a few areas. The Reissner beam is one of the simplest acoustical system that can be treated in the context of mechanics with symmetry. It seems that the non-linear phenomena can be handled in their intrinsic qualities through the concepts of differential geometry. Using the symmetry of Lie groups, the geometric constructions needed for reduction are presented in the context of the "covariant" approach.**

INTRODUCTION

This article is devoted to extend, in the context of field theories, some results of symplectic geometry.

A. Article organization

In order to state the subject, the study made by Elie Cartan on variational problems is related in a brief subsection. He has explained, in his "Leçon sur les invariants intégraux" [1], the way the Poincaré-Cartan form is obtained and what are its properties. The differential of the Poincaré-Cartan form, called pre-symplectic form, gives rise to the Hamilton's equation of motion and can be related to the Poisson formalism.

After this historical introduction a section is dedicated to extend the discussion to a more general jet-bundle and leads to what is called now *multi-symplectic geometry*[1]. In this context, the proofs are more laborious but follow the main ideas of Cartan. This leads to a general formalism (section I) where theories on reduction by Lie group action can be handled with confidence in the second part of this article (section II). Hamiltonian formalism finishes this paper.

B. Historical remark: Cartan's lesson

Let's summarize Elie Cartan's lesson where Hamiltonian formalism is obtained by introducing in a natural way the Poincaré-Cartan form. This form comes from considering variations of the action functional along real trajectories with variable boundary conditions. The action functional is written $\mathcal{A} = \int_{t_0}^{t_1} \pounds(q, \dot{q}, t)\,\mathrm{d}t$, (where $\pounds(q, \dot{q}, t)$ is the lagrangian density of the system).

Cartan expresses the variation of the action $\delta\mathcal{A}$ using a variation vector field $Z(\varepsilon, t)$. With his "magic rule" for the Lie derivative of a form: $\mathsf{L}_Z\alpha = Z \lrcorner\, \mathrm{d}\alpha + \mathrm{d}(Z \lrcorner\, \alpha)$, he may compute (integrating by part)

$$\begin{aligned}\delta\mathcal{A} &= \mathrm{d}\mathcal{A}(Z) = \left[\frac{\partial \pounds}{\partial \dot{q}}\,\frac{\partial q_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \pounds\,\mathrm{d}t(Z)\right]_{t_0(\varepsilon)}^{t_1(\varepsilon)} \\ &- \int_{t_0(\varepsilon)}^{t_1(\varepsilon)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \pounds}{\partial \dot{q}}\right) - \frac{\partial \pounds}{\partial q}\right)\,\frac{\partial q_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0}\mathrm{d}t.\end{aligned} \quad (1)$$

By choosing a vector field of variation, Z, vanishing on the boundaries t_0, t_1, the first term disappears. The principle of least action then leads to the well known Euler-Lagrange equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \pounds}{\partial \dot{q}}\right) - \frac{\partial \pounds}{\partial q} = 0. \quad (2)$$

But Elie Cartan continues his discussion by choosing variable boundary conditions (the boundaries $t_i(\varepsilon)$ depend on a parameter of variation ε). With this assumption and for real trajectories verifying the Euler-Lagrange equation (2), the variation of action reduces to the first term of (1) since the second integral vanishes. On the boundaries, $t_i(\varepsilon)$, he then uses the relation

$$\frac{\partial q_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \mathrm{d}q(Z) - \dot{q}\mathrm{d}t(Z), \quad (3)$$

in (1) to obtain $\delta\mathcal{A} = [\Theta]_{t_0}^{t_1}$ introducing the Poincaré-Cartan form

$$\Theta = p\,\mathrm{d}q - \mathcal{H}dt, \quad (4)$$

with new variables $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ and $\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}$: the Legendre transform appears very naturally.

Then, Cartan shows that this computation leads to an integral invariant along critical sections (solutions) of the variational problem. To do so, he considers a collection of real trajectories labeled by a parameter ε. *"Finally, suppose that we consider a tube of trajectories, i.e., a closed continuous linear collection of trajectories, each of which is limited to a time interval (t_0, t_1) that varies also with ε. The formula which gives the variation of the action along these variable trajectories reduces to*

$$\delta\mathcal{A} = [\Theta]_{t_0}^{t_1} = (\Theta)_1 - (\Theta)_0.$$

[1] By the way, it is worth mentioning that Cartan in 1933 in [2] thought about doing a geometry where geodesics would be replaced by (hyper)surfaces.

When one returns to the initial trajectory the total variation of the action is obviously zero, in such a way that, if one integrates with respect to ε then one will have

$$\oint \delta\mathcal{A} = 0 \Rightarrow \oint (\Theta)_1 = \oint (\Theta)_0$$

[...] *given an arbitrary tube of trajectories, if the integral $\oint \Theta$ is taken along a closed curve around the tube then that integral will be independent of that curve and will depend only upon the tube..."*

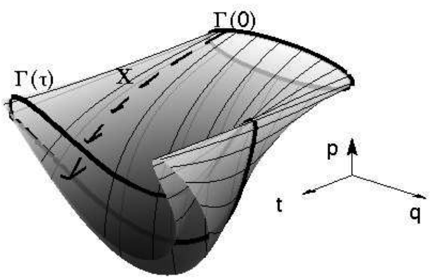

Fig. 1. Tube of real trajectories: the integral $I = \oint_\Gamma \Theta$ does not depend on the choice of the closed loop Γ around the tube. This quantity is thus an integral invariant that depends only on the choice of the tube.

So, the quantity $I = \oint \Theta$ is invariant i.e. does not depend on the closed curve Γ along the tube of trajectories. In a modern language, I is invariant along the vector field X tangent to the critical sections, that is

$$0 = \mathrm{d}I(X) = \oint_\Gamma \mathsf{L}_X \Theta = \oint_\Gamma X \lrcorner \mathrm{d}\Theta + \oint_{\partial\Gamma} X \lrcorner \Theta.$$

The last integral taken over a closed loop is obviously null. Since this property is true for any choice of the tube, one must have $X \lrcorner \mathrm{d}\Theta = 0$. So Cartan introduces the differential of the Poincaré-Cartan from: $\Omega = -\mathrm{d}\Theta = \mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}\mathcal{H} \wedge \mathrm{d}t$ that verifies $X \lrcorner \Omega = 0$, for all vector field X tangent to real trajectories. In other words, since Ω is also a closed form, the Lie derivative of the (pre)-symplectic form vanishes

$$\mathsf{L}_X \Omega = 0, \quad \forall X \text{ tangent to real trajectories.} \tag{5}$$

He then notices that each coefficient of $\delta q = \mathrm{d}q(X)$, $\delta p = \mathrm{d}p(X)$ and $\delta t = \mathrm{d}t(X)$ in $X \lrcorner \Omega = 0$, i.e.

$$(\mathrm{d}p + \frac{\partial \mathcal{H}}{\partial q}\mathrm{d}t)\delta q - (\mathrm{d}q - \frac{\partial \mathcal{H}}{\partial p}\mathrm{d}t)\delta p - (\mathrm{d}\mathcal{H} - \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t)\delta t = 0$$

must necessarily be canceled. Doing so, he recovers the Hamilton canonical equation of motion

$$\begin{cases} \mathrm{d}q = \frac{\partial \mathcal{H}}{\partial p}\mathrm{d}t \\ \mathrm{d}p = -\frac{\partial \mathcal{H}}{\partial q}\mathrm{d}t \\ \mathrm{d}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t \end{cases} \tag{6}$$

C. Poisson formalism

More generally, the invariance (5) of the (pre)-symplectic form along real trajectories verifying $dt(X) = 1$ can be written $X \lrcorner \mathrm{d}q \wedge \mathrm{d}p = \mathrm{d}\mathcal{H}$. The real trajectories can be viewed as a canonical transformation (an infinitesimal transformation that conserves the canonical symplectic structure $\tilde{\Omega} = \mathrm{d}q \wedge \mathrm{d}p$). It gives rise to the notion of hamiltonian vector field Y_F associated to a canonical transformation F defined by

$$Y_F \lrcorner \tilde{\Omega} = \mathrm{d}F.$$

This last statement can also be written $\tilde{\Omega}(Y_F, .) = \mathrm{d}F(.)$. Thus, considering the Hamiltonian vector field $X_\mathcal{H}$ associated to a physical problem, the dynamic of any observable F can be computed by $\Omega(Y_F, X_\mathcal{H}) = \mathrm{d}F(X_\mathcal{H})$ where the last term is related to the variation of F along real trajectories: that is $dF(X_\mathcal{H}) = \frac{\mathrm{d}F}{\mathrm{d}t} = \dot{F}$. This leads, by introducing the Poisson bracket $\{F, \mathcal{H}\} = \tilde{\Omega}(Y_F, X_\mathcal{H})$, to the Poisson equation

$$\dot{F} = \{F, \mathcal{H}\}. \tag{7}$$

I. Lagrangian formalism of first-order field theories

This discussion has started in 1922 in a context where only one independent variable "t" was taken into account in variational problems. In modern language, one would speak about *symplectic geometry*. The considered space is a jet-bundle with unidimensional base (i.e. endowed with volume form $\omega = dt$).

This section relates this formalism to a more general jet-bundle. In the context of field theory, the state space (t, q, v) (time, position, velocity) originally conceived by Cartan to study the geometry of the trajectories (curves) according to the optimization principle, is extended to fiber bundles. To be more precise, let M be an orientable manifold and $\pi : E \to M$ a differentiable fiber bundle with typical fibre F (dim $F = N$). In the fiber bundle context, more than one independent variable (dim $M = n+1$) are allowed for (space-time) parametrization and then, the concept of curves is generalized to the concept of sections ($\phi : M \to E$ in the sequel).

To take into account the velocities, it is also necessary to lift these sections to the extended jet-bundle j^1E (velocity space for short) to obtain $(j^1\phi) : M \to j^1E$. The holonomic concept is used to that purpose with the help of a *contact form* θ which might be related to the form (3) in Cartan's lesson. Furthermore, an optimization process seeks the critical section among a family of variated sections generated by a infinitesimal transformation. The vector field tangent to this transformation (say Z) needs also to be lifted by a holonomic process (j^1Z in the sequel).

A. Principle of least action

From now on, M is an oriented manifold and $\omega \in \Lambda^{n+1}(M)$ is a fixed volume (n + 1)-form on M. With these elements well-defined (see Arturo Echeverria-Enriquez & al [3] and appendix A1 for notations) the lagrangian formalism is used. The lagrangian form is written as

$$\mathcal{L} = \pounds(x^\mu, y^A, v^A_\mu)\omega, \quad \pounds \in \mathscr{C}^\infty(J^1E),$$

in a natural local system (x^μ, y^A, v^A_μ) on J^1E. So, we can define

DEFINITION I.1 Hamilton principle
Let $((E, M; \pi), \mathcal{L})$ be a lagrangian system. Let $\Gamma_c(M, E)$ be the set of compact supported sections of π and consider the (action) map

$$\begin{aligned} \mathcal{A} : \Gamma_c(M,E) &\rightarrow \mathbb{R} \\ \phi &\mapsto \int_M (j^1\phi)^*\mathcal{L}. \end{aligned}$$

The variational problem posed by the lagrangian density $\mathcal{L}$ is the problem of searching for the critical (or stationary) sections of the functional $\mathcal{A}$.

The section ϕ must be stationary with respect to variations given by $\phi_\epsilon = \tau_\epsilon \circ \phi$, where τ_ϵ is a local one-parameter group of any vector field $Z \in \chi^{V(\pi)}(E)$ which is π-vertical, i.e. $Z = \beta^A \vec{\partial}_A$ (with lift (33), see appendix for details). Nevertheless, even in this more complicated circumstance, a similar expression to (1) may be obtained to express the variation of the action functional, $\forall \boldsymbol{\beta}$

$$\delta\mathcal{A} = \int_{\partial M} (j^1\phi)^* j^1 Z \lrcorner \Theta_{\mathcal{L}} \qquad (8)$$
$$- \int_M (\beta^A \circ \phi) \left[\vec{\partial}_\mu \left(\frac{\partial \pounds}{\partial v^A_\mu} \circ (j^1\phi)\right) - \left(\frac{\partial \pounds}{\partial y^A} \circ (j^1\phi)\right) \right] \omega.$$

Now, the (Lagrangian) Poincaré-Cartan $(n+1)$-form

$$\Theta_{\mathcal{L}} = \frac{\partial \pounds}{\partial v^A_\mu} dy^A \wedge d^n x_\mu - \left(\frac{\partial \pounds}{\partial v^A_\mu} v^A_\mu - \pounds \right) \omega, \qquad (9)$$

is introduced with the n-form $d^n x_\mu = \vec{\partial}_\mu \lrcorner \omega$. Following Cartan's idea, the Poincaré-Cartan (n+1-form) is obtained from the variation of the action functional and the Legendre transformation[2] follows in a natural way as it appears clearly in (9).

On one hand, choosing a vector field Z that vanishes on the boundary ∂M gives the Euler-Lagrange field equations

$$\vec{\partial}_\mu \left.\frac{\partial \pounds}{\partial v^A_\mu}\right|_{(j^1\phi)} - \left.\frac{\partial \pounds}{\partial y^A}\right|_{(j^1\phi)} = 0, \quad A = 1, \ldots, N. \qquad (10)$$

On the other hand, if Z is not compactly supported, the first integral on the boundary ∂M in the variation (8) must be taken into account. Choosing sections ϕ verifying the Euler-Lagrange equations (10), the second integral vanishes. The lagrangian pre-multisymplectic $(n+2)$-form in J^1E, $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$, is then used in the variation theorem (see, for example, [3] for proof and details).

B. Variation theorem

[2]Legendre transformation $p^\mu_A = \frac{\partial \pounds}{\partial v^A_\mu}$, $\mathcal{H} = \frac{\partial \pounds}{\partial v^A_\mu} v^A_\mu - \pounds$.

THEOREM 1 Variation theorem
The following assertions regarding a section ϕ of the bundle $\pi : E \rightarrow M$ are equivalent
(i) ϕ is a stationary point of $\mathcal{A} = \int_M (j^1\phi)^*\mathcal{L}$;
(ii) the Euler–Lagrange equations (10) hold in coordinates;
(iii) for any vector field X on J^1E

$$(j^1\phi)^*(X \lrcorner \Omega_{\mathcal{L}}) = 0. \qquad (11)$$

The variation theorem is really useful especially in presence of symmetry. It allows to demonstrate the first Noether theorem and to obtain a conserved quantity named *current*.

C. Lagrangian symmetries and Noether's theorem

In Mechanics, a symmetry of a lagrangian dynamical system is a diffeomorphism in the phase space of the system (the tangent bundle) which leaves the lagrangian function invariant. It can be thought of being generated by a vector field. This leads to the following definition:

DEFINITION I.2 An infinitesimal natural symmetry
Let $((E, M; \pi), \mathcal{L})$ be a lagrangian system. An infinitesimal natural symmetry of the lagrangian system is a vector field $S \in \chi(E)$ such that its canonical prolongation leaves $\mathcal{L}$ invariant (vanishing Lie derivative)

$$\mathsf{L}_{(j^1S)}\mathcal{L} = 0.$$

If the vector field $S \in \chi(E)$ is an infinitesimal natural symmetry of the lagrangian system $((E, M; \pi), \mathcal{L})$ then the Poincaré-Cartan form is also invariant, i.e $\mathsf{L}_{j^1S}\Theta_{\mathcal{L}} = 0$.

According to the first Noether's theorem, the presence of symmetries leads to conserved quantities. The main result is:

THEOREM 2 First Noether's theorem
Let $S \in \chi(E)$ be an infinitesimal natural symmetry of the lagrangian system $((E, M; \pi), \mathcal{L})$. Then, the n-form $J(S) := (j^1S) \lrcorner \Theta_{\mathcal{L}}$ is constant (closed) on the critical sections of the variational problem posed by $\mathcal{L}$.

Proof: let $\phi : M \rightarrow E$ be a critical section of the variational problem, that is, according to the variation theorem (11)

$$(j^1\phi)^* X \lrcorner d\Theta_{\mathcal{L}} = 0, \quad \forall X \in \chi(J^1E)$$

Since $\mathcal{L}$ is invariant under S, we also have the invariance of the Poincaré-Cartan form

$$0 = \mathsf{L}_{j^1S}\Theta_{\mathcal{L}} = d\left(j^1S \lrcorner \Theta_{\mathcal{L}}\right) + j^1S \lrcorner d\Theta_{\mathcal{L}}.$$

Therefore, on critical section

$$\begin{aligned} 0 &= (j^1\phi)^*\mathsf{L}_{j^1S}\Theta_{\mathcal{L}} \\ &= (j^1\phi)^*\left[d\left(j^1S \lrcorner \Theta_{\mathcal{L}}\right)\right] + (j^1\phi)^*\left[j^1S \lrcorner d\Theta_{\mathcal{L}}\right] \\ &= (j^1\phi)^*\left[d\left(j^1S \lrcorner \Theta_{\mathcal{L}}\right)\right] = d\left[(j^1\phi)^*\left(j^1S \lrcorner \Theta_{\mathcal{L}}\right)\right] \end{aligned}$$

and the result follows. For critical section ϕ of the variational problem posed by $\mathcal{L}$, the expression $(j^1\phi)^*J(S)$ is called *Noether's current* associated with S.

II. Lagrangian formalism with symmetry

To obtain the preceding results, the vector field of variation (say Z) needs to be lifted (j^1Z). This process can be handled by considering the canonical contact form given in coordinates by $\boldsymbol{\theta} = (\mathrm{d}y^A - v^A_\mu \mathrm{d}x^\mu) \otimes \vec{\partial}_A$ (see appendix A2 and A3).

What happens if we now consider a principal bundle where the fiber is a Lie group? What are the expressions of the contact form and the lift of vector field in this context? What does become of the Euler-Lagrange equations of motion and what about the Poincaré-Cartan form? If all these questions had an answer then the Noether theorem will give new expression of an invariant current along solutions.

Since each velocity can be translated to the tangent space at the identity e of the group G, the *Lie algebra* $\mathfrak{g}$ is used in the 1-jet bundle definition and the specific canonical contact form must be expressed using the Maurer-Cartan form. The definition generally used to compute the lift of vector field j^1Z is to say that its flow leaves invariant the contact module $\boldsymbol{\theta}$. We would prefer, for practical reasons, a more geometric definition saying that a holonomic section stay holonomic under the action of a contact transformation.

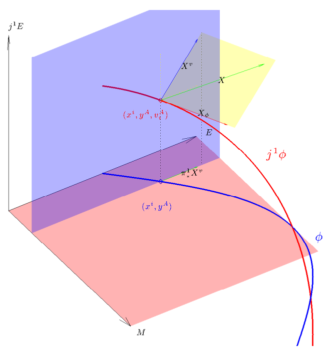

Fig. 2. The section $(j^1\phi)$ is called the canonical lifting or the canonical prolongation of ϕ to J^1E. A section of $\jmath\pi$ which is the canonical extension of a section of π is called a **holonomic section**

Thus, in the following (and it is the main interest of this article), the Poincaré-Cartan and multi-symplectic forms are obtained for a principal G bundle. It allows to formulate the *Euler-Poincaré equations* of motion and leads to a Noether current defined in the dual Lie algebra.

A. Principal bundle

1) Left-Right invariant bases and canonical contact form: Let consider now a principal G bundle $\pi : E \to M$ with structure group G. A point in the bundle is $\mathrm{P} = (x^\mu, g^A)$ where $\boldsymbol{g}$ belongs to the group G ($\mu = 1, \dots, n, 0$ and $A = 1, \dots, N$). Let $\vec{e}_A$, be the left-invariant basis on TG obtained by left translation $TL_{\boldsymbol{g}}$ from the identity e of the vectors $\vec{\partial}_A$

$$\vec{e}_A = T_e L_{\boldsymbol{g}}(\vec{\partial}_A)$$

If the point $\bar{\mathrm{P}} = (x^\mu, g^A, \xi^A_\mu)$ of the 1-jet bundle J^1E is over $\mathrm{P} = (x^\mu, g^A)$ then it exists a section $\boldsymbol{\phi}$ representative of that point such that $\xi^A_\mu = \lambda^A(\frac{\partial \boldsymbol{\phi}}{\partial x^\mu})$. Here, the form λ^A is the Maurer-Cartan 1-form: dual basis of the left-invariant basis $\vec{e}_A$ defined by $\lambda^A(\vec{e}_B) = \delta^A_B$. The computation of the contact form gives in this context

$$\boldsymbol{\theta}_{\bar{\mathrm{P}}} = (\lambda^A - \xi^A_\mu dx_\mu) \otimes \vec{e}_A. \tag{12}$$

B. Jet prolongation of vector fields: the lift

If a vector field is given in the coordinate system (x^μ, g^A) by

$$Z = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{e}_A, \tag{13}$$

its lift requires the variation of the Maurer-Cartan form $\boldsymbol{\lambda} = \lambda^A \otimes \vec{\partial}_{\xi^A}$ which is involved in the contact form (12). The idea to obtain this jet-prolongation (or the infinitesimal contact transformation) is

- to start from a holonomic section $(j^1\phi)$,
- to transform $\phi : M \to E$ into $\phi_\epsilon = \tau^Z_\epsilon \circ \phi$ according to the flow generated by Z,
- to ask for the new section $(j^1\phi_\epsilon)$ to be also holonomic.

Doing this in an infinitesimal way, the jet-prolongation j^1Z of Z is obtained.

If the section $\phi : M \to E$ is characterized by $n+1$ tangent vector fields X_μ ($\mu = 1, \dots, n, 0$), this infinitesimal procedure will involve the bracket of vector fields. More precisely, the Lie derivative of the two vector fields X and Z, at point P, is given by (see [4] for example),

$$\mathsf{L}_Z X = [Z, X]\,(\mathrm{P}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left((\tau^Z_\epsilon)^* X(\mathrm{P}')\right)$$
$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left(T_{(\tau^Z_{-\epsilon})} X(\mathrm{P}')\right) = \lim_{\epsilon \to 0} \frac{T_{(\tau^Z_{-\epsilon})} X(\mathrm{P}') - X(\mathrm{P})}{\epsilon}$$

where $\mathrm{P}' = \tau^Z_\epsilon(\mathrm{P})$. Evaluating this definition at point P', one obtains a finite expansion of the tangent map of the flow τ^Z_ϵ

$$X_\epsilon(\mathrm{P}') = T_{(\tau^Z_\epsilon)} X(\mathrm{P}) = X(\mathrm{P}') + \epsilon\,[X, Z]\,(\mathrm{P}') + \mathcal{O}(\epsilon^2) \tag{14}$$

which represents also the vector field tangent to the transformed section ϕ_ϵ. Without loss of generality, it is convenient to consider the "normalized" vector fields

$$\tilde{X}_\mu = \vec{\partial}_\mu + \xi^A_\mu \vec{e}_A \tag{15}$$

which are tangent to the section ϕ at point (x^μ, g^A, ξ^A_μ). Since the lifted section $(j^1\phi)$ (resp. $(j^1\phi_\epsilon)$) is asked to be holonomic, the contact form (12) taken in the direction of $\tilde{X}_\mu$ cancels and gives the following relation

$$\lambda^A(\tilde{X}_\mu) = \xi^A_\nu dx^\nu(\tilde{X}_\mu) = \xi^A_\nu \delta^\nu_\mu = \xi^A_\mu$$

(resp. $\lambda^A_\epsilon(\tilde{X}^\epsilon_\mu) = (\xi_\epsilon)^A_\mu$). So the lifts of the sections, $(j^1\phi)$ and $(j^1\phi)_\epsilon$, go respectively through the points (x^μ, g^A, ξ^A_μ) and $(x^\mu_\epsilon, g^A_\epsilon, (\xi_\epsilon)^A_\mu)$. The component along $\vec{\partial}_{\xi^A}$ of j^1Z (the lift of the vector field Z) is then obtained by the following calculus

$$d\xi^A_\mu(j^1Z) = \lim_{\epsilon \to 0} \frac{(\xi_\epsilon)^A_\mu - \xi^A_\mu}{\epsilon} = \lim_{\epsilon \to 0} \frac{\lambda^A_\epsilon(\tilde{X}^\epsilon_\mu) - \lambda^A(\tilde{X}_\mu)}{\epsilon}.$$

This expression involves the variation of the Maurer-Cartan form λ_ϵ^A in the Z direction that can be again expressed using the Lie derivative formula $\left(\mathsf{L}_Z\lambda^A\right)(X) = \lim_{\epsilon\to 0}\frac{\lambda^A_{\mathrm{P}_\epsilon}(X_\epsilon)-\lambda^A_{\mathrm{P}}(X)}{\epsilon}$ to give

$$\lambda^A_{\mathrm{P}_\epsilon}(X_\epsilon) = \lambda^A_{\mathrm{P}}(X) + \epsilon\left(\mathsf{L}_Z\lambda^A\right)(X) + \mathcal{O}(\epsilon^2). \qquad (16)$$

The fact that the section ϕ_ϵ, transformed under the action of Z with tangent vectors $\tilde{X}_i^\epsilon$, is also holonomic is equivalent to the following linear system

$$\boldsymbol{\theta}_{\bar{\mathrm{P}}}(\tilde{X}_i^\epsilon) = 0 \Rightarrow \lambda_\epsilon^A(\tilde{X}_i^\epsilon) = (\xi_\epsilon)^A_\nu dx^\nu(\tilde{X}_i^\epsilon).$$

From this and using finite expansions (14) and (16) for vector fields and Maurer-Cartan form, one can easily obtain

$$(\xi_\epsilon)^A_\mu = \xi^A_\mu + \epsilon\left(\left(\mathsf{L}_Z\lambda^A\right)(\tilde{X}_\mu) - \xi^A_i dx^i(\left[\tilde{X}_\mu, Z\right])\right) + \mathcal{O}(\epsilon^2)$$

and thus

$$d\xi^A_\mu(j^1Z) \quad = \quad \left(\mathsf{L}_Z\lambda^A\right)(\tilde{X}_\mu) - \xi^A_i dx^i(\left[\tilde{X}_\mu, Z\right])$$

This can be simplified in three steps.

1) First step: Using the vectorial 1-form $\boldsymbol{\lambda}$ and noting that, on one hand, $\boldsymbol{\lambda}_{\mathrm{P}}(\tilde{X}_\mu) = \boldsymbol{\xi}_\mu$ and that, on the other hand taking into account the Maurer-Cartan equation for $\boldsymbol{\lambda}$, we have

$$\begin{aligned}
&\left(\mathsf{L}_Z\boldsymbol{\lambda}\right)(\tilde{X}_\mu) = \left(d(Z\lrcorner\boldsymbol{\lambda}) + (Z\lrcorner d\boldsymbol{\lambda})\right)(\tilde{X}_\mu)\\
=\;& \left(d\boldsymbol{\beta} - (Z\lrcorner[\boldsymbol{\lambda},\boldsymbol{\lambda}])\right)(\tilde{X}_\mu) \quad \text{since } d\boldsymbol{\lambda} = -[\boldsymbol{\lambda},\boldsymbol{\lambda}]\\
=\;& \left(d\boldsymbol{\beta} - ([\boldsymbol{\lambda}(Z),\boldsymbol{\lambda}])\right)(\tilde{X}_\mu)\\
=\;& d\boldsymbol{\beta}(\tilde{X}_\mu) + \left[\boldsymbol{\lambda}(\tilde{X}_\mu),\boldsymbol{\lambda}(Z)\right]\\
=\;& d\boldsymbol{\beta}(\tilde{X}_\mu) + \left[\boldsymbol{\xi}_\mu,\boldsymbol{\beta}\right], \quad \left(= d\boldsymbol{\beta}(\tilde{X}_\mu) + \boldsymbol{\lambda}\left[\tilde{X}_\mu, Z\right]\right)
\end{aligned}$$

and then

$$d\xi^A_\mu(j^1Z) \quad = \quad d\beta^A(\tilde{X}_\mu) + \underbrace{\lambda^A(\left[\tilde{X}_\mu, Z\right]) - \xi^A_i dx^i(\left[\tilde{X}_\mu, Z\right])}_{\theta^A_{\bar{\mathrm{P}}}(\left[\tilde{X}_\mu, Z\right])}$$

2) Second step: As the differential of the map $(x^\mu, g^A)\mapsto \vec{e}_A$ has no component along $\vec{\partial}_\mu$, i.e. $d(\vec{e}_A) = \Omega^B_A\vec{e}_B$, the computation of the commutator

$$\begin{aligned}
&\left[\tilde{X}_\mu, Z\right] = dZ(\tilde{X}_\mu) - d\tilde{X}_\mu(Z)\\
=\;& \left(d\alpha^\nu\vec{\partial}_\nu + d\beta^A\vec{e}_A + \beta^A d(\vec{e}_A)\right)(\tilde{X}_\mu) - \left(\xi^A_\mu d(\vec{e}_A)\right)(Z)\\
=\;& d\alpha^\nu(\tilde{X}_\mu)\vec{\partial}_\nu + \left[(d\beta^A + \beta^A\Omega^B_A)(\tilde{X}) - \xi^A_\mu\Omega^B_A(Z)\right]\vec{e}_B
\end{aligned}$$

shows[3] that the term $dx^i(\left[\tilde{X}_\mu, Z\right])$ equals $d\alpha^i(\tilde{X}_\mu)$. So we have now

$$d\xi^A_\mu(j^1Z) = d\beta^A(\tilde{X}_\mu) + \left[\boldsymbol{\xi}_\mu,\boldsymbol{\beta}\right]^A - \xi^A_i d\alpha^i(\tilde{X}_\mu).$$

[3]It is important to notice that $\tilde{X}_\mu$ are defined for a fixed ξ^A_μ.

3) Third step: Expressing any vector $\tilde{X}_\mu$ in both basis $\vec{e}_A$ and $\vec{\partial}_A$ gives $\tilde{X}_\mu = \vec{\partial}_\mu + \xi^I_\mu\vec{e}_I = \vec{\partial}_\mu + v^Q_\mu\vec{\partial}_Q$, for some ξ^I_μ and v^Q_μ. That is, if dy is the dual basis to the basis $\vec{\partial}_A$,

$$\begin{cases}\lambda^A(\tilde{X}_\mu) & = \xi^A_\mu = \lambda^A(v^Q_\mu\vec{\partial}_Q) = v^Q_\mu\lambda^A(\partial_{y^Q})\\ dy^B(\tilde{X}_\mu) & = v^B_\mu = dy^B(\xi^I_\mu\vec{e}_I) = \xi^I_\mu dy^B(\vec{e}_I)\end{cases}$$

and then since β^A is a function of x and y

$$\begin{aligned}
d\beta^A(\tilde{X}_\mu) &= \left(\frac{\partial\beta^A}{\partial x^\nu}dx^\nu + \frac{\partial\beta^A}{\partial y^B}dy^B\right)\left(\vec{\partial}_\mu + \xi^I_\mu\vec{e}_I\right)\\
&= \frac{\partial\beta^A}{\partial x^\mu} + \frac{\partial\beta^A}{\partial y^B}\xi^I_\mu dy^B(\vec{e}_I) = \left(\frac{\partial\beta^A}{\partial x^\mu} + v^B_\mu\frac{\partial\beta^A}{\partial y^B}\right)
\end{aligned}$$

(same computation for α). Finally, the lift of a vector field given by (13), at point $\bar{\mathrm{P}} = (x^\mu, g^A, \xi^A_\mu)$ is

$$\begin{aligned}
d\xi^A_\mu(j^1Z)_{\bar{\mathrm{P}}} \quad = \quad & \left(\frac{\partial\beta^A}{\partial x^\mu} + v^B_\mu\frac{\partial\beta^A}{\partial y^B}\right) - \xi^A_\nu\left(\frac{\partial\alpha^\nu}{\partial x^\mu} + v^B_\mu\frac{\partial\alpha^\nu}{\partial y^B}\right)\\
& + \quad \left[\boldsymbol{\xi}_\mu,\boldsymbol{\beta}\right]^A,
\end{aligned}$$

remembering that $v^B_\mu = dy^B(\tilde{X}_\mu) = \frac{\partial\phi^B}{\partial x^\mu}$. It mainly differs by a Lie bracket term from the standard formalism (32) given for convenience in appendix A3. if $Z\in\chi^{V(\pi)}(E)$ is a vertical vector field (that is, $\pi_*Z = 0$ - then if τ_ϵ is a local one-parameter group associated with Z, it induces the identity on M), it would be written $Z = \beta^A\vec{e}_A$, in a local natural system of coordinates, with canonical prolongation (lift)

$$j^1Z = \beta^A\vec{e}_A + \left[\left(\frac{\partial\beta^A}{\partial x^\mu} + v^B_\mu\frac{\partial\beta^A}{\partial y^B}\right) + \left[\boldsymbol{\xi}_\mu,\boldsymbol{\beta}\right]^A\right]\vec{\partial}_{\xi^A_\mu}. \qquad (17)$$

III. Lagrangian reduction formalism of first-order field theories

This section is devoted to the introduction of reduced lagrangian densities. The reduction is due to symmetry induced by the action of a lie Group G. Following, step by step, the construction made in the section I for standard lagrangian formalism, the geometrical objects (Poincaré-Cartan form and Euler-Poincaré equation) are then obtained naturally and allow us to study the dynamical behaviour of field theories with symmetry. The reduced lagrangian form is now written as

$$\mathfrak{L} = \mathbb{l}(x^\mu, g^A, \xi^A_\mu)\omega, \quad \mathbb{l}\in\mathscr{C}^\infty(J^1E),\ \omega\in\Lambda^{n+1}(M)$$

in a natural local system (x^μ, g^A, ξ^A_μ) on J^1E. This reduced lagrangian is used in the Hamilton principle I.1 with the action functional $\mathcal{A} = \int_M (j^1\phi)^*\mathfrak{L}$.

A. Euler Poincaré equations and Poincaré-Cartan form

Introducing the co-adjoint operator ad^* such that

$$\left(\left[\boldsymbol{\xi}_\mu,\boldsymbol{\beta}\right],\boldsymbol{\pi}\right) = \left(\mathrm{ad}_{\boldsymbol{\xi}_\mu}\boldsymbol{\beta},\boldsymbol{\pi}\right) = \left(\boldsymbol{\beta},\mathrm{ad}^*_{\boldsymbol{\xi}_\mu}\boldsymbol{\pi}\right),$$

the same procedure as in preceding sections can be used. In the computation of $\delta\mathcal{A}$ (i.e. introducing the lift formula 17), the (Lagrangian) Poincaré-Cartan $(n{+}1)$-form must be written

$$\Theta_{\mathfrak{L}} = \frac{\partial\mathbb{l}}{\partial\xi^A_\mu}\lambda^A\wedge d^nx_\mu - \left(\frac{\partial\mathbb{l}}{\partial\xi^A_\mu}\xi^A_\mu - \mathbb{l}\right)\omega, \qquad (18)$$

to obtain the same form as (1) or (8). That is, $\forall \boldsymbol{\beta}$,

$$\begin{aligned}\delta\mathcal{A} &= \int_{\partial M}(j^1\phi)^* j^1 Z \lrcorner\, \Theta_{\mathfrak{L}} \qquad (19)\\ &- \int_M (\beta^A\circ\phi)\Big[\vec{\partial}_\mu\big(\frac{\partial \mathbb{l}}{\partial \xi^A_\mu}\circ(j^1\phi)\big)\\ &- (T_{AB}\frac{\partial \mathbb{l}}{\partial y^B}\circ(j^1\phi)) - \left(\mathrm{ad}^*_{\boldsymbol{\xi}_\mu}\frac{\partial \mathbb{l}}{\partial \boldsymbol{\xi}_\mu}\right)^A\circ(j^1\phi)\Big]\,\omega,\end{aligned}$$

setting $T_{AB} = dy^B(\vec{e}_A)$. It obviously furnishes the equations of motion, named Euler-Poincaré equations, $\forall A = 1,\ldots,N$

$$\vec{\partial}_\mu \frac{\partial \mathbb{l}}{\partial \xi^A_\mu}\bigg|_{(j^1\phi)} - \left(\mathrm{ad}^*_{\boldsymbol{\xi}_\mu}\frac{\partial \mathbb{l}}{\partial \boldsymbol{\xi}_\mu}\right)^A\bigg|_{(j^1\phi)} - T_{AB}\frac{\partial \mathbb{l}}{\partial y^B}\bigg|_{(j^1\phi)} = 0. \qquad (20)$$

The Lagrangian Poincaré-Cartan $(n+2)$-form (or pre-multisymplectic) in J^1E is then introduced as usual by $\Omega_{\mathfrak{L}} = -d\Theta_{\mathfrak{L}}$.

B. Legendre transformation

The covariant Legendre transformation for $\mathfrak{L}$ is now constructed. It is a fiber-preserving map between the Jet-bundle and its dual $\mathbb{F}\mathfrak{L} : J^1E \to J^1E^*$ over E which has the coordinate expressions

$$\pi^A_\mu = \frac{\partial \mathbb{l}}{\partial \xi^A_\mu}, \quad \mathbb{h} = \frac{\partial \mathbb{l}}{\partial \xi^A_\mu}\xi^A_\mu - \mathbb{l} \qquad (21)$$

for the multimomenta π^A_μ and the covariant reduced hamiltonian $\mathbb{h}$. In this circumstance, the Cartan form $\Theta_{\mathfrak{L}}$ (resp. $\Omega_{\mathfrak{L}}$) appears to be the pulling back of a canonical form Θ_h (resp. Ω_h) on J^1E^*

$$\Theta_{\mathfrak{L}} = \mathbb{F}\mathfrak{L}^*\Theta_h \quad (\text{resp. } \Omega_{\mathfrak{L}} = \mathbb{F}\mathfrak{L}^*\Omega_h)$$

that is

$$\Theta_h = \pi^A_\mu \lambda^A \wedge d^n x_\mu - \mathbb{h}\omega, \text{ and } \Omega_h = -d\Theta_h \qquad (22)$$

C. Symmetries for reduced Lagrangian systems

1) Infinitesimal symmetries: For the particular case where the group G acts on itself, by a left action

$$\begin{aligned} L_m : G &\to G\\ \boldsymbol{g} &\mapsto m\circ\boldsymbol{g},\end{aligned}$$

an infinitesimal generator can be defined. Lets take a curve $m(s) = \exp(\boldsymbol{\eta}s)$ through the identity at $s=0$ with tangent vector $\boldsymbol{\eta}$. It gives rise to the curve $\boldsymbol{g}(s) = m(s)\circ\boldsymbol{g}$ and, from a section ϕ, to a family of sections $\phi_s = m(s)\circ\phi$. By definition, the infinitesimal generator vector field of the left action, at point $\boldsymbol{g}$, is given by $S_\eta = \frac{d}{ds}\Big|_{s=0}\boldsymbol{g}(s)$ which is the definition of the tangent map of the right action R_g since

$$\frac{d}{ds}\bigg|_{s=0}\boldsymbol{g}(s) = \frac{d}{ds}\bigg|_{s=0}(m(s)\circ\boldsymbol{g}) = \frac{d}{ds}\bigg|_{s=0}R_g m(s) = T_gR(\boldsymbol{\eta}).$$

In other words, S_η is the right invariant vector field X^R_η generated by $\boldsymbol{\eta}\in\mathfrak{g}$. It coincides, at any point $\boldsymbol{g}$, with a left invariant vector field X^L_ψ where $\boldsymbol{\psi}$ is related to $\boldsymbol{\eta}$ by the adjoint operator

$$\boldsymbol{\eta} = \mathrm{Ad}_g\boldsymbol{\psi}, \quad \text{or } \boldsymbol{\psi} = \mathrm{Ad}_{g^{-1}}\boldsymbol{\eta}$$

2) Noether current: If an infinite natural symmetry, S_η, leaves the reduced lagrangian invariant (i.e. $\mathsf{L}_{j^1S_\eta}\mathfrak{L} = 0$), the Noether's theorem 2 may be applied. That is to say that a Noether current $J_\eta = j^1S_\eta \lrcorner\, \Theta_{\mathfrak{L}}$ can be associated to each $\boldsymbol{\eta}\in\mathfrak{g}$ using the Poincaré-Cartan form $\Theta_{\mathfrak{L}}$. The question is then to compute the prolongation j^1S_η from S_η. Since S_η is a right invariant vector field, it has constant coordinates on the right basis

$$S_\eta = \eta^A\vec{e}^{\,\mathrm{R}}_A = X^R_\eta,$$

(from now on, right and left invariant basis are denoted by $\vec{e}^{\,\mathrm{R}}$ and $\vec{e}^{\,\mathrm{L}}$). So, in the left basis $\vec{e}^{\,\mathrm{L}}_A$ we have

$$S_\eta = X^L_\psi = \psi^A\vec{e}^{\,\mathrm{L}}_A = (\mathrm{Ad}_{g^{-1}}\boldsymbol{\eta})^A\vec{e}^{\,\mathrm{L}}_A,$$

according to the fact that $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ are related by the adjoint operator. The prolongation j^1S_η can then be written, for some γ^A_μ,

$$j^1S_\eta = (\mathrm{Ad}_{g^{-1}}\boldsymbol{\eta})^A\vec{e}^{\,\mathrm{L}}_A + \gamma^A_\mu\vec{\partial}_{\xi^A_\mu}.$$

Its contraction with the Poincaré-Cartan form $\Theta_{\mathfrak{L}}$ is then

$$\begin{aligned} J_\eta &= j^1S_\eta \lrcorner\, \Theta_{\mathfrak{L}}\\ &= \frac{\partial \mathbb{l}}{\partial \xi^A_\mu}\lambda^A(j^1S_{\boldsymbol{\eta}})\wedge d^nx_\mu - \left(\frac{\partial \mathbb{l}}{\partial \xi^A_\mu}\xi^A_\mu - \mathbb{l}\right)\omega(j^1S_\eta)\\ &= \frac{\partial \mathbb{l}}{\partial \xi^A_\mu}(\mathrm{Ad}_{g^{-1}}\boldsymbol{\eta})^A\wedge d^nx_\mu = \left(\pi^\mu_A, (\mathrm{Ad}_{g^{-1}}\boldsymbol{\eta})^A\right)d^nx_\mu\\ &= \left((\mathrm{Ad}^*_{g^{-1}}\boldsymbol{\pi}^\mu)^A, \eta^A\right)d^nx_\mu = (\boldsymbol{\Pi}^\mu, \boldsymbol{\eta})\,d^nx_\mu \quad \forall\boldsymbol{\eta}\end{aligned}$$

It allows to define a Noether current n-form

$$J = \boldsymbol{\Pi}^\mu d^nx_\mu \qquad (23)$$

by stating $J_\eta = (J, \boldsymbol{\eta})$. Here we recognize the right momentum $\boldsymbol{\Pi}^\mu = \mathrm{Ad}^*_{g^{-1}}\boldsymbol{\pi}^\mu$ expressed from the left momentum $\pi^\mu_A = \frac{\partial \mathbb{l}}{\partial \xi^A_\mu}$. The n-form J is constant (closed) on the critical sections of the variational problem posed by $\mathfrak{L}$ which gives the balance law

$$d[(j^1\phi)^*J] = (j^1\phi)^*dJ = (j^1\phi)^*\big(\frac{\partial \boldsymbol{\Pi}^\mu}{\partial x^\mu}\big)\,\omega = 0. \qquad (24)$$

3) Example: Noether current for the Reissner's beam: In that case, $J = \boldsymbol{\Pi}^\mu d^nx_\mu = \Pi^s d^nx_s + \Pi^t d^nx_t$, and $\omega = ds\wedge dt$. So,

$$\begin{aligned} d^nx_s &= \partial_s \lrcorner\,(ds\wedge dt) = dt\\ d^nx_t &= \partial_t \lrcorner\,(ds\wedge dt) = -ds\\ \Pi^s &= \boldsymbol{\Sigma} = \mathrm{Ad}^*_{g^{-1}}\boldsymbol{\sigma}_L, \quad \boldsymbol{\sigma}_L = \mathbb{C}\boldsymbol{\epsilon}_L\\ \Pi^t &= \boldsymbol{\Pi} = \mathrm{Ad}^*_{g^{-1}}\boldsymbol{\pi}_L, \quad \boldsymbol{\pi}_L = \mathbb{H}\boldsymbol{\nu}_L\end{aligned}$$

which gives the Noether current

$$J = \boldsymbol{\Sigma}dt - \boldsymbol{\Pi}ds$$

and the balance law

$$0 = (j^1\phi)^*dJ = (j^1\phi)^*\left(\frac{\partial \boldsymbol{\Sigma}}{\partial s} + \frac{\partial \boldsymbol{\Pi}}{\partial t}\right)\omega.$$

IV. Hamiltonian formalism

A. Covariant or multisyplectic Hamiltonian formalism

In this section, the covariant (or multisymplectic) hamiltonian formalism is developed (see also [5]).

1) Elements for the covariant hamiltonian formalism: Following Marsden [6], we introduce sections in the dual space J^1E^* by the definition

DEFINITION IV.1
L et ϕ be a section of the fiber bundle $\pi : E \to M$ and $(j^1\phi)$ its first jet. A section $(j^1\phi)^*$ of J^1E^* is called conjugate to $(j^1\phi)$ if

$$(j^1\phi)^* = \mathbb{F}\mathfrak{L} \circ (j^1\phi).$$

In this case we say that $(j^1\phi)^*$ is holonomic.

With this definition, the variation theorem 1 is modified to

THEOREM 3
I f the Legendre transformation $\mathbb{F}\mathfrak{L} : J^1E \to J^1E^*$ is a fiber diffeomorphism over E, the following assertions regarding a section ϕ of the bundle $\pi : E \to M$ are equivalent

(i) ϕ is a stationary point of $\int_M (j^1\phi)^*\mathcal{L}$;
(ii) $(j^1\phi)^*$ is a Hamiltonian section for $\mathcal{H}$, that is to say that for any vector field W on J^1E^*

$$(j^1\phi)^*(W \lrcorner \Omega_{\mathcal{H}}) = 0. \tag{25}$$

2) De Donder-Weyl equations for Hamiltonian with symmetry: Let's calculate the expression $W \lrcorner \Omega_h = 0$ according to the multisymplectic form Ω_h. From (22) one obtains

$$W \lrcorner \left(- \left(d\pi^A_\mu \wedge \lambda^A - \pi^A_\mu \left[\boldsymbol{\lambda}, \boldsymbol{\lambda}\right]^A \right) \wedge d^n x_\mu + d\mathbb{h} \wedge \omega \right) = 0.$$

The pull-back by $(j^1\phi)^*$ yields the De Donder-Weyl equations

$$\begin{cases} \frac{\partial \pi^A_\mu}{\partial x_\mu}\Big|_{(j^1\phi)^*} - (\mathrm{ad}^*_{\frac{\partial \mathbb{h}}{\partial \boldsymbol{\pi}^\mu}} \boldsymbol{\pi}^\mu)^A\Big|_{(j^1\phi)^*} + T_{AB}\frac{\partial \mathbb{h}}{\partial y^A}\Big|_{(j^1\phi)^*} = 0 \\ \xi^A_\mu\Big|_{(j^1\phi)^*} = \frac{\partial \mathbb{h}}{\partial \pi^A_\mu}\Big|_{(j^1\phi)^*}, \end{cases} \tag{26}$$

that is to say, the hamiltonian form of Euler-Poincaré equation (20) (note that the last equation is the inverse Legendre transformation). Right invariant halmitonian form can also be obtained as (change of sign in the adjoint term)

$$\begin{cases} \frac{\partial \Pi^A_\mu}{\partial x_\mu}\Big|_{(j^1\phi)^*} + (\mathrm{ad}^*_{\frac{\partial \mathbb{h}'}{\partial \boldsymbol{\Pi}^\mu}} \boldsymbol{\Pi}^\mu)^A\Big|_{(j^1\phi)^*} + T^R_{BA}\frac{\partial \mathbb{h}'}{\partial y^A}\Big|_{(j^1\phi)^*} = 0 \\ \chi^A_\mu\Big|_{(j^1\phi)^*} = \frac{\partial \mathbb{h}'}{\partial \Pi^A_\mu}\Big|_{(j^1\phi)^*}. \end{cases} \tag{27}$$

where the Hamiltonian $\mathbb{h}'$ is expressed with right multimomentum $\boldsymbol{\Pi}$ and right velocity $\boldsymbol{\chi} \in \mathfrak{g}$.

B. Hamiltonian form of Noether conservation law

In the sequel, it is convenient to introduce the right invariant Maurer-Cartan form ρ. That is, If the point $\bar{\mathrm{P}} = (x^\mu, g^A, \chi^A_\mu)$ of the 1-jet bundle J^1E is over $\mathrm{P} = (x^\mu, g^A)$ then it exists a section $\boldsymbol{\phi}$ representative of that point such that $\chi^A_\mu = \rho^A(\frac{\partial \boldsymbol{\phi}}{\partial x^\mu})$.

1) Symmetric vector field: If now, we consider a vector field of symmetry. To be more precise, if the lagrangian density is invariant under the left action of a Lie group, we consider a right invariant vector field $S_\eta = \eta^A \vec{e}^{\,R}_A$ with constant vector $\boldsymbol{\eta}$ in the Lie-algebra $\mathfrak{g}$. This vector field has, according to (17), an extension[4] at the point $(x^\mu, y^A, \chi^A_\mu) \in J^1E$

$$j^1 S_\eta = \eta^A \vec{e}^{\,R}_A - \left[\boldsymbol{\chi}_\mu, \boldsymbol{\eta}\right]^A \partial_{\chi^A_\mu} \tag{28}$$

By symmetry the Lie derivative of the reduced lagrangian ($\mathbb{l}'$ express on the right for example) vanishes: $\mathsf{L}_{j^1S_\eta}\mathbb{l}' = 0$. This means $d\mathbb{l}'(j^1S_\eta) = 0$, since j^1S is a π-vertical vector field (no component on $\vec{\partial}_\mu$).

2) Conservation law: So, we have by symmetry and with change of basis [5]

$$\begin{aligned} 0 &= d\mathbb{l}'(j^1S_\eta) = \left(\frac{\partial \mathbb{l}'}{\partial x^\mu} dx^\mu + \frac{\partial \mathbb{l}'}{\partial y^A} dy^A + \frac{\partial \mathbb{l}'}{\partial \chi^A_\mu} d\chi^A_\mu \right)(j^1S_\eta) \\ &= \frac{\partial \mathbb{l}'}{\partial y^A} dy^A(j^1S_\eta) + \frac{\partial \mathbb{l}'}{\partial \chi^A_\mu} d\chi^A_\mu(j^1S_\eta) \\ &= \frac{\partial \mathbb{l}'}{\partial y^B} dy^B(\partial_{y^A_R})\rho^A(j^1S_\eta) + \frac{\partial \mathbb{l}'}{\partial \chi^A_\mu} d\chi^A_\mu(j^1S_\eta) \\ &= \frac{\partial \mathbb{l}'}{\partial y^B} T^R_{AB}\rho^A(j^1S_\eta) - \frac{\partial \mathbb{l}'}{\partial \chi^A_\mu}\left[\boldsymbol{\chi}_\mu, \boldsymbol{\eta}\right]^A, \quad \text{by lift (28)} \\ &= \frac{\partial \mathbb{l}'}{\partial y^B} T^R_{AB}\eta^A - \Pi^A_\mu \left[\boldsymbol{\chi}_\mu, \boldsymbol{\eta}\right]^A, \quad \text{by Legendre transf.} \\ &= \left(-\frac{\partial \mathbb{h}'}{\partial y^B} T^R_{AB} - (\mathrm{ad}^*_{\boldsymbol{\chi}_\mu}\boldsymbol{\Pi}^\mu)^A \right)\eta^A, \quad \text{since } \frac{\partial \mathbb{l}'}{\partial y^B} = -\frac{\partial \mathbb{h}'}{\partial y^B} \end{aligned}$$

for all $\boldsymbol{\eta}$, that is $(\mathrm{ad}^*_{\boldsymbol{\chi}_\mu}\boldsymbol{\Pi}^\mu)^A = -\frac{\partial \mathbb{h}'}{\partial y^B}T^R_{AB}$. So these two terms in the de Donder equation (27) annihilate each other to give the conservation law $\frac{\partial \Pi^A_\mu}{\partial x_\mu}\Big|_{(j^1\phi)^*} = 0$ which was all ready obtained before in (24). It appears that, with left-invariant lagrangians, the first Noether theorem 2 can be formulated by the right formulation of the Hamilton-Poincaré equation of motion (27).

3) Noether's current:

a) Left: The Noether's current is computed by contracting the Hamiltonian version (22) of the Poincaré-Cartan form with a left expression of the symmetric vector field

$$j^1 S_\eta = \mathrm{ad}_{g^{-1}}\boldsymbol{\eta}^A \vec{e}^{\,L}_A + \gamma^A_\mu \partial_{\xi^A_\mu}$$

[4] the minus sign is due to the right invariance and the formula (17) is for left invariance.

[5] Change of dual basis. If $V = \beta^B \vec{\partial}_B = \alpha^B \vec{e}^{\,R}_B$, we have $dy^A(V) = \beta^A = dy^A(\alpha^B \vec{e}^{\,R}_B) = dy^A(\vec{e}^{\,R}_B)\alpha^B = dy^A(\vec{e}^{\,R}_B)\rho^B(V), \forall V$. Thus, $dy^A = dy^A(\vec{e}^{\,R}_B)\rho^B$.

for some γ. It gives

$$\begin{aligned} j^1 S_\eta \lrcorner \Theta_h &= \left(\pi^A_\mu \lambda^A \wedge d^n x_\mu - \mathbb{h}\omega\right)(j^1 S_\eta) \\ &= \pi^A_\mu (\mathrm{ad}_{g^{-1}}\boldsymbol{\eta})^A d^n x_\mu = \left(\boldsymbol{\pi_\mu}, \mathrm{ad}_{g^{-1}}\boldsymbol{\eta}\right) d^n x_\mu \\ J^*_{L\,\eta} &= \left(\mathrm{ad}^*_{g^{-1}}\boldsymbol{\pi}^\mu, \boldsymbol{\eta}\right) d^n x_\mu = (J^*_L, \boldsymbol{\eta}) \end{aligned}$$

That is

$$J^*_L = \mathrm{ad}^*_{g^{-1}}\boldsymbol{\pi}^\mu d^n x_\mu \tag{29}$$

b) Right: The right version is obtained by contracting the Hamiltonian Poincaré-Cartan form with a right expression of the symmetric vector field (28). It gives

$$\begin{aligned} j^1 S_\eta \lrcorner \Theta_{h'} &= \left(\Pi^A_\mu \rho^A \wedge d^n x_\mu - \mathbb{h}'\omega\right)(j^1 S_\eta) \\ &= \Pi^A_\mu \eta^A d^n x_\mu \\ J^*_{R\,\eta} &= \left(\boldsymbol{\Pi}^\mu, \boldsymbol{\eta}\right) d^n x_\mu = (J^*_R, \boldsymbol{\eta}). \end{aligned}$$

Which is the same as (23), i.e.

$$J^*_R = \boldsymbol{\Pi}^\mu d^n x_\mu. \tag{30}$$

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Appendix

A. Elements of differential geometry for first-order lagrangian field theories

1) Geometrical structures of first-order jet bundles: For all the article, the framework of Arturo Echeverria-Enriquez & al [3] is adopted as a system of notation. It is quickly related here for convenience of the reader. Let M be an orientable manifold and $\pi : E \to M$ a differentiable fiber bundle with typical fibre F. Let dim $M = n+1$, dim $F = N$. The bundle of 1-jets of sections of π, or 1-jet bundle, is denoted by J^1E. For every $\mathrm{P} \in E$, the fiber of J^1E is denoted $J^1_{\mathrm{P}}E$ and its elements by $\bar{\mathrm{P}}$. If $\phi : U \subset M \to E$ is a representative of $\bar{\mathrm{P}} \in J^1_{\mathrm{P}}E$, we write $\bar{\mathrm{P}} = T_{\pi(\mathrm{P})}\phi$.

Sections of π can be lifted to J^1E in the following way: let x^μ, $\mu = 1, \dots, n, 0$, be a local system in M and y^A, $A = 1, ..., N$ a local system in the fibers; that is, $\{x^\mu, y^A\}$ is a coordinate system adapted to the bundle. In these coordinates, a local section $\phi : U \to E$ is written as $\phi(x) = (x^\mu, \phi^A(x))$, that is, $\phi(x)$ is given by functions $y^A = \phi^A(x)$.

This local system $(x^\mu,\, y^A)$ allows us to construct a local system (x^μ, y^A, v^A_μ) in J^1E, where v^A_μ are defined as follows: if $\bar{\mathrm{P}} \in J^1E$, with $\pi^1(\bar{\mathrm{P}}) = \mathrm{P}$ and $\pi(\mathrm{P}) = x$, let $\phi : U \to E$, $y^A = \phi^A$, be a representative of $\bar{\mathrm{P}}$, then

$$v^A_\mu(\bar{\mathrm{P}}) = \left(\frac{\partial \phi^A}{\partial x^\mu}\right)\Big|_x$$

These systems are called natural local systems in J^1E. In one of them, we have

$$(j^1\phi)(x) = (x^\mu(x), \phi^A(x), \frac{\partial \phi^A}{\partial x^\mu}(x)).$$

2) Canonical form and holonomy: The bundle J^1E is endowed with a canonical geometric structure θ with expression in a natural local system

$$\boldsymbol{\theta}_{\bar{\mathrm{P}}} = (\mathrm{d}y^A - v^A_\mu \mathrm{d}x^\mu) \otimes \vec{\partial}_A \tag{31}$$

Holonomic sections can be characterized using this canonical form as follows:

Proposition A.1
L et $\psi : M \to J^1E$ be a section of $\jmath\pi$. The necessary and sufficient condition for ψ to be a holonomic section is that $\psi^*\theta = 0$.

3) Jet prolongation of vector fields : A local diffeomorphism $\varphi : J^1E \to J^1E$ defines a contact transformation if it preserves the contact ideal, meaning that if σ is any contact form on J^1E, then $\varphi^*\sigma$ is also a contact form. The flow generated by a vector field j^1Z on the jet space J^1E forms a one-parameter group of contact transformations if and only if the Lie derivative $\mathsf{L}_{J^1Z}(\sigma)$ of any contact form σ preserves the contact ideal or module.

So, starting from a general vector field $Z = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A$, where α^μ and β^Adepend on $(x^\mu,\, y^A)$, and writing the jet prolongation j^1Z on the jet space J^1E as

$$j^1Z = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A + \gamma^A_\mu \vec{\partial}^\mu_A,$$

the only problem is to calculate the coefficients γ^A_μ.

Echeverria & al [3] traduce the preservation of the contact module generated by the forms $\theta^A = (\mathrm{d}y^A - v^A_\nu \mathrm{d}x^\nu)$ by the fact that they must have $\mathsf{L}_{J^1Z}(\theta^A) = \zeta^A_B \theta^B$ for some $\mathscr{C}^\infty$ function ζ^A_B on J^1E. So, using the Cartan formula in the Lie derivative L_{J^1Z} and identifying the different terms, they obtain the 1-jet prolongation of Z with coefficients

$$\gamma^A_\mu = \frac{\partial \beta^A}{\partial x^\mu} + v^B_\mu \frac{\partial \beta^A}{\partial y^B} - v^A_\nu \left(\frac{\partial \alpha^\nu}{\partial x^\mu} + v^B_\mu \frac{\partial \alpha^\nu}{\partial y^B}\right). \tag{32}$$

As a particular case, if $Z \in \chi^{V(\pi)}(E)$ is a π-vertical vector field, in a local natural system of coordinates, Z is equal to $\beta^A \vec{\partial}_A$ and its canonical prolongation is

$$j^1Z = \beta^A \vec{\partial}_A + \left(\frac{\partial \beta^A}{\partial x^\mu} + v^B_\mu \frac{\partial \beta^A}{\partial y^B}\right) \vec{\partial}^\mu_A. \tag{33}$$