Energy-balanced models for acoustic and audio systems: a port-Hamiltonian approach†

Antoine Falaize, Nicolas Lopes and Thomas Hélie
IRCAM - CNRS: UMR 9912
1, place Igor Stravinsky
75004 Paris - France
Email: {falaize, lopes, helie}@ircam.fr

Denis Matignon
Université de Toulouse - ISAE
10, av. Édouard Belin - BP 54032
31055 Toulouse Cedex 4 - France
Email: denis.matignon@isae.fr

Bernhard Maschke
LAGEP - CNRS: UMR5007
43, Bd. du 11 Novembre 1918
69622 Villeurbanne Cedex - France
Email: maschke@lagep.univ-lyon1.fr

***Abstract*—Port-Hamiltonian systems (PHS) are an extension of Hamiltonian systems, which represent open passive systems which are structured according to their conservative and dissipative parts, as well as the external sources. They are widely used in engineering as a central tool for the modelling of physical systems, their passive-guaranteed simulation, as well as for control issues, see e.g. [1]. In this paper, we shall recall the definition of the port-Hamiltonian systems and give some indication on how they might be used for the modelling and simulation of electro-acoustic systems, which are illustrated on some applications.**

INTRODUCTION

A key point in the simulation and control of audio systems is the preservation of passivity, which implies stability. This paper focuses on the so-called *Port-Hamiltonian Systems* formalism, which encodes the power balance. The first section illustrates the basic concepts of the finite-dimensional port-Hamiltonian structure with the direct relevant benefits of this framework with respect to passive modeling.

Section II is a reminder on port-Hamiltonian systems, which presents them as an extension of systems of conservation laws in infinite and finite dimension to open systems. We start by consider Hamiltonian systems of conservation laws and show how one may define a Dirac structure encompassing interface variables at the boundary and thereby define Hamiltonian systems allowing for energy flows through the boundary of their spatial domain [2]. Secondly, we shall recall how Dirac structure may be derived from the topological structure of the system such as graphs of circuits [3] and shall show how they may be used to consider open circuits, allowing for boundary currents or voltage determined from the interaction with its environment. Thirdly we briefly recall how port Hamiltonian systems may be used to encompass dissipative relations which are essential in acoustic applications.

In section III, different applications under current development shall be presented. The first application is the derivation of a Hamiltonian model of a non-linear electro-acoustic transducer. A structure-preserving time-discretization scheme which guarantees passivity and, as consequence, the stability of the simulations is presented section III-A. The second application in section III-B is the derivation of a simple but energetically consistent model of the coupling between an artificial lip (the musician), a jet and an acoustic tube (the instrument). Section III-C is concerned with the port-Hamiltonian formulation of finite dimensional mechanical systems including structural damping of Caughey type. An extension to infinite dimensional systems is proposed in section III-D, with application to the Euler-Bernouilli beam.

† The contribution of the authors has been done within the context of the French National Research Agency sponsored project HAMECMOPSYS. Further information is available at http://www.hamecmopsys.ens2m.fr/.

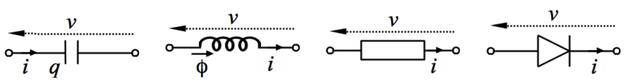
I. MOTIVATIONS AND BASIC EXAMPLES

Most physical systems prove passive. They are made of storage components with total energy $\mathtt{E}(t) > 0$, dissipative components with dissipated power $P_{\mathcal{D}}(t) > 0$ and external ports with total provided power $P_{\mathcal{S}}(t)$. Port-Hamiltonian systems (PHS) encode this passivity property through the power balance $\frac{\mathtt{dE}(t)}{\mathtt{d}t} = -P_{\mathcal{D}}(t) + P_{\mathcal{S}}(t)$. The *Hamiltonian* is the total energy $\mathcal{H} : \mathbf{x} \mapsto \mathcal{H}(\mathbf{x}) \in \mathbb{R}_+$ expressed by a chosen state $\mathbf{x} \in \mathbb{R}^{n_S}$ so that $\frac{\mathtt{dE}}{\mathtt{d}t} = \nabla\mathcal{H}(\mathbf{x})^T \frac{\mathtt{dx}}{\mathtt{d}t}$. The dissipated power is factorized as $P_{\mathcal{D}} = \mathbf{z}(\mathbf{w})^T\mathbf{w}$ with $\mathbf{w} \in \mathbb{R}^{n_D}$ an extra chosen state and $\mathbf{z} : \mathbb{R}^{n_D} \to \mathbb{R}^{n_D}$ a dissipation function. A similar factorization is done for external sources with input vector $\mathbf{u}\in\mathbb{R}^{n_P}$ and output vector $\mathbf{y}\in\mathbb{R}^{n_P}$, so that $P_{\mathcal{S}} = \mathbf{u}^T\mathbf{y}$.

We now illustrate these concepts on the basic elements of figure 1. The electrostatic (potential) energy stored in the capacitor of figure 1(a) is $\mathcal{H}_C(x_C) = \frac{x_C^2}{2C}$ with the charge $x_C = q$. Then $\frac{\mathtt{d}\mathcal{H}_C}{\mathtt{d}x_C} = v$ and $\frac{\mathtt{d}x_C}{\mathtt{d}t} = i$. Choosing $u := i$ as input, and $y := v$ as output, the power received by the capacitor is $P_{\mathcal{S}} = uy$ and

$$\begin{pmatrix} \frac{\mathtt{d}x_C}{\mathtt{d}t} \\ -y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathbf{S}_C} \cdot \begin{pmatrix} \frac{\mathtt{d}\mathcal{H}_C}{\mathtt{d}x_C} \\ u \end{pmatrix}. \quad (1)$$

This encodes the power balance since $\mathbf{S}_C$ is skew-symetric: $(\frac{\mathtt{d}\mathcal{H}_C}{\mathtt{d}x_C}, u) \cdot (\frac{\mathtt{d}x_C}{\mathtt{d}t}, -y)^T = 0 = \frac{\mathtt{d}\mathcal{H}_C}{\mathtt{d}t} - P_{\mathcal{S}}$. Similarly for the inductor 1(b), the electrodynamic (inertial) enregy is $\mathcal{H}_L(x_L) = \frac{x_L^2}{2L}$ with the magnetic flux $x_L = \phi$. Then $\frac{\mathtt{d}\mathcal{H}_L}{\mathtt{d}x_L} = i$ and $\frac{\mathtt{d}x_L}{\mathtt{d}t} = v$. Choosing now $u := v$ as input, and $y := i$ as output, the power received by the inductor is $P_{\mathcal{S}} = uy$ and structure (1) holds with the power balance $(\frac{\mathtt{d}\mathcal{H}_L}{\mathtt{d}x_L}, u) \cdot (\frac{\mathtt{d}x_L}{\mathtt{d}t}, -y)^T = \frac{\mathtt{d}\mathcal{H}_L}{\mathtt{d}t} - P_{\mathcal{S}} = 0$.



(a) Capacitor (b) Inductor (c) Resistance (d) Diode

Fig. 1. Elementary components

For the resistance (figure 1(c)), we define the state $w = i$ and the dissipative characteristic $z(w) = Rw$, with dissipated power $P_{\mathcal{D}} = wz(w)$. Again, choosing $u := i$ as input, and $y := v$ as output, the power received by the resistance is $P_{\mathcal{S}} = uy$ and structure (1) still holds:

$$\begin{pmatrix} w \\ -y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathbf{S}_R} \cdot \begin{pmatrix} z(w) \\ u \end{pmatrix} \qquad (2)$$

with the power balance $(z(w), u) \cdot (w, -y)^T = P_{\mathcal{D}} - P_{\mathcal{S}} = 0$ ($\mathbf{S}_R$ skew-symetric). For the diode (figure 1(d)), we define the state $w = v$ and the non-linear dissipative characteristic $z(w) = \mathrm{I}_s(e^{w/v_0} - 1)$ (Shockley equation) with dissipated power $P_{\mathcal{D}} = wz(w)$. Choosing $u := w$ as input, and $y := i$ as output, the power received by the diode is $P_{\mathcal{S}} = uy$, and structure (2) holds with exactly the same power balance as in the linear case.

Generally speaking, we focus on physical systems (mechanic, electronic, magnetic, chemical, etc.) governed by some energetic principles and well-posed power balance. In this paper, we are interested in such systems known as *port-Hamiltonian systems* (PHS), associated to *Dirac structures*. Especially, we are interested in systems which can be explicitly described by the structure

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \\ \hline \mathbf{w} \\ \hline -\mathbf{y} \end{pmatrix}}_{A} = \underbrace{\left(\begin{array}{c|c|c} \mathbf{J_x} & -\mathbf{K} & \mathbf{G_x} \\ \hline \mathbf{K}^T & \mathbf{J_w} & \mathbf{G_w} \\ \hline -\mathbf{G_x}^T & -\mathbf{G_w}^T & \mathbf{J_y} \end{array}\right)}_{\mathbf{S}} \cdot \underbrace{\begin{pmatrix} \nabla\mathcal{H}(\mathbf{x}) \\ \hline \mathbf{z}(\mathbf{w}) \\ \hline \mathbf{u} \end{pmatrix}}_{B} \qquad (3)$$

with $\mathbf{J_x} \in \mathbb{R}^{n_S \times n_S}$, $\mathbf{J_w} \in \mathbb{R}^{n_D \times n_D}$, $\mathbf{J_y} \in \mathbb{R}^{n_P \times n_P}$ skew-symmetric matrices[1]. Structure (3) restores the power balance, since $\mathbf{S}$ becomes skew-symetric: $B^T A = \frac{\mathrm{dE}}{\mathrm{d}t} + P_{\mathcal{D}} - P_{\mathcal{S}} = 0$. Hence, (3) proves to be a PHS.

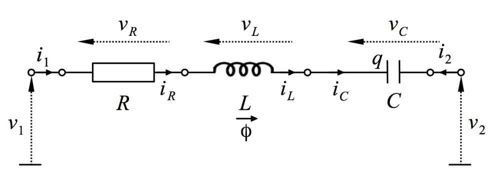


Fig. 2. A simple RLC circuit. q is the charge on one of the two plates of the capacitor with capacitance C and ϕ is the magnetic flux considered in the solenoid with inductance L. A linear resistor is denoted by R.

As an illustration, consider the simple *RLC*-circuit of figure 2. Applying Kirchhoff's laws, the dynamic of this circuit can be recast as the PHS (3), with definitions of table I. Note that $\nabla\mathcal{H}(\mathbf{x}) = (\nu_C, i_L)^T$ and $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = (i_C, \nu_L)^T$.

[1]That is, $\mathbf{J}^T = -\mathbf{J}$.

$\mathbf{x} = (\ q, \phi\ )^T$	$\mathcal{H}(\mathbf{x}) = \mathcal{H}_C(x_1) + \mathcal{H}_L(x_2)$
$w = i_R$	$z(w) = Rw$
$\mathbf{J_x} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{K} = \begin{pmatrix} 0 \\ +1 \end{pmatrix}$, $\mathbf{G_x} = \begin{pmatrix} 0 & 0 \\ +1 & -1 \end{pmatrix}$ Other matrices are 0 with appropriate dimensions.	

TABLE I
STORAGE $\mathcal{H}$, DISSIPATION z AND STRUCTURE $\mathbf{S}$ OF THE RLC CIRCUIT OF FIGURE 2. THE CHARGE IS q AND THE MAGNETIC FLUX IS ϕ. INPUT/OUTPUT QUANTITIES ARE $\mathbf{u} := (\nu_1, \nu_2)^T$ AND $\mathbf{y} := (i_1, i_2)^T$.

II. Reminder on Port Hamiltonian systems

Port Hamiltonian systems are extension of Hamiltonian systems which encompass some set of external (or interface) variables allowing to formulate models of *open* physical systems. This class of systems has been introduced in [4] as an extension of Hamiltonian systems defined on Poisson manifolds, and then found numerous applications in Dynamical systems' theory and engineering, see e.g. [1], [5]. However in this section, we shall firstly present the port Hamiltonian systems from the point of view of systems of conservation laws in infinite dimension [2], and secondly in finite dimension [3]. Thirdly we shall extend this definition from conservative to dissipative physical systems.

A. Port Hamiltonian systems derived from systems of conservation laws

In this paragraph, for sake of simplicity, we shall introduce port Hamiltonian stemming from a Hamiltonian system of two conservation laws according to [1, chap.4] . This construction may be generalized similarly to an arbitrary number of conservation laws as it may be found for instance in multiscale systems [6], fluid dynamics [7] or thermo-magneto-hydrodynamic models of plasma for instance [8].

The state space of the dynamical systems considered hereafter are exterior differential forms [9], for which we recall some notations. We consider a spatial domain being an n-dimensional oriented connected open smooth submanifold denoted by $Z \subset \mathbb{R}^n$ of dimension k and denote by $\Omega^k(Z)$ the set of *exterior differential forms of degree* k on Z (called hereafter k*-forms*). Denote by $\Omega = \bigoplus_{k \geq 0} \Omega^k(Z)$ the algebra of differential forms over Z. It is endowed with an *exterior product* $\wedge$, which endows Ω with a structure of graded algebra and the *exterior derivation* (also called *coboundary map*) denoted by d. Finally we shall use the *Hodge star product*, denoted by $\star$ associated with Euclidean structure of $\mathbb{R}^n$. Furthermore we shall use a *nondegenerate pairing* between k-forms of complementary degree $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ given by

$$< \omega^{n-k} | \omega^k >:= \int_Z \omega^{n-k} \wedge \omega^k \quad (\in \mathbb{R}), \qquad (4)$$

with $\omega^k \in \Omega^k(Z)$, $\omega^{n-k} \in \Omega^{n-k}(Z)$.

Finally we shall use an adapted definition of the *variational derivative* of a functional

$$H(\alpha) := \int_Z \mathcal{H}(\alpha) \in \mathbb{R}, \tag{5}$$

with density $\mathcal{H} : \Omega^p(Z) \times Z \to \Omega^n(Z)$, with respect to the differential form $\alpha \in \Omega^p(Z)$, as the uniquely defined differential form $\frac{\delta H}{\delta \alpha} \in \Omega^{n-p}(Z)$ such that

$$H(\alpha + \varepsilon \Delta\alpha) = \int_Z \mathcal{H}(\alpha) + \varepsilon \int_Z \left[\frac{\delta H}{\delta \alpha} \wedge \Delta\alpha \right] + \mathrm{O}\left(\varepsilon^2\right)$$

for any $\alpha, \Delta\alpha \in \Omega^p(Z)$ with compact support strictly included in Z and $\varepsilon \in \mathbb{R}$.

Let us consider a system of two conservation laws on some *conserved quantities* $\alpha_i \in \Omega^{k_i}(Z),\ i \in \{1, 2\}$

$$\frac{\partial \alpha_i}{\partial t} + \mathrm{d}\beta_i = g_i \tag{6}$$

where $\beta_i \in \Omega^{k_i - 1}(Z)$ denote the set of *fluxes* and $g_i \in \Omega^{k_i}(Z)$ denote the set of *distributed source forms*.

In the sequel, we shall consider closure relations for the fluxes which describe the canonical interaction of two physical domains (for instance involving kinetic energy and elastic energy in elasto-dynamics or electric and magnetic energies for electromagnetic fields) and are generated by a Hamiltonian functional (5)

$$\begin{pmatrix} \beta_p \\ \beta_q \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \tag{7}$$

where $r = p\,q + 1$ and $\varepsilon \in \{-1, +1\}$ depending on sign convention on the physical domain. Combining equations (6) and (7), the system of two conservation laws may be written as an infinite-dimensional Hamiltonian system

$$\frac{\partial \alpha}{\partial t} = \mathcal{J} \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \tag{8}$$

defined with respect to the *matrix differential operator*:

$$\mathcal{J} = \varepsilon \begin{pmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \tag{9}$$

which is *Hamiltonian* (i.e. skew-symmetric and satisfy the Jacobi identities, hence defines a *Poisson bracket*) under the condition that the variational derivatives $\frac{\delta H}{\delta \alpha_p}$ and $\frac{\delta H}{\delta \alpha_q}$ have compact support in the spatial domain [10]. If this is not the case, meaning in terms of physical modelling that there is some flow of energy flowing through the boundary of the domain, the matrix operator $\mathcal{J}$ is no more a Hamiltonian operator.

However one may still define a Hamiltonian system by extending the Hamiltonian operator $\mathcal{J}$ to a Dirac structure. Dirac structures [11], [12], [13] have been introduced as generalizations of Poisson brackets and presymplectic structures and are defined as follows.

Definition 1. Let $\mathscr{F}$ and $\mathscr{E}$ be two real vector spaces and assume that they are endowed with a *non degenerate bilinear form* denoted by:

$$\begin{array}{rcl} \langle . | . \rangle : \mathscr{F} \times \mathscr{E} & \to & \mathbb{R} \\ (f,\, e) & \mapsto & \langle e | \, f \rangle \end{array} \tag{10}$$

The bilinear product leads to the definition of a *symmetric bilinear form* on the product space $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ as follows:

$$\begin{array}{rcl} \ll \cdot, \cdot \gg : \mathscr{B} \times \mathscr{B} & \to & \mathbb{R} \\ ((f_1,\, e_1),\, (f_{,2}\; e_2)) & \mapsto & \ll (f_1, e_1), (f_2, e_2) \gg \\ & & = \langle e_1 | \, f_2 \rangle + \langle e_2 | \, f_1 \rangle \end{array} \tag{11}$$

A *Dirac structure* is a linear subspace $\mathscr{D} \subset \mathscr{B}$ such that $\mathscr{D} = \mathscr{D}^{\perp}$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\ll, \gg$.

The matrix differential operator in (9) will be extended by defining interface variables, called *boundary port variables*, as the restriction, denoted by $\left.\right|_{\partial Z}$, of the flux variables to the boundary ∂Z of the domain Z:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \beta_q|_{\partial Z} \\ \beta_p|_{\partial Z} \end{pmatrix} \tag{12}$$

This defines the following Dirac structure associated with the differential operator (9), called *Stokes-Dirac structure*.

Proposition 2. *Consider the space of flow variables* $\mathcal{F}_{p,q}$

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z) \ni (f_p, f_q, f_\partial) \tag{13}$$

and the space of effort variables $\mathcal{E}_{p,q}$

$$\mathcal{E}_{p,q} := \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z) \ni (e_p, e_q, e_\partial) \tag{14}$$

with the integers p *and* q *satisfying* $p + q = n + 1$. *Define* $r := pq + 1$. *The linear subspace* D *of the bond space* $\mathcal{B}_{p,q} = \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$*:*

$$\begin{aligned} D \;=\; & \Big\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \,| \\ & \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \varepsilon \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ & \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \Big\} \end{aligned} \tag{15}$$

is a Dirac structure.

The proof of the proposition may be found in [2] and the name Stokes-Dirac structure has been chosen because its definition essentially relies on the use of Stokes' theorem.

Using this definition of Stokes-Dirac structure, system (8) with boundary port variables (12) may be defined as boundary port-Hamiltonian system defined with respect to the Stokes-Dirac structure.

Definition 3. The boundary *port-Hamiltonian system of two conservation laws* with n-dimensional manifold of spatial variables Z, state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p + q = n + 1$ and $r = pq + 1$), Stokes-Dirac structure D given by (15) and Hamiltonian functional H, is defined as follows

$$\left(-\frac{\partial \alpha_p}{\partial t},\, -\frac{\partial \alpha_q}{\partial t},\, f_\partial, \delta_p H,\, \delta_q H,\, e_\partial \right) \in D.$$

The classical example are the Maxwell's equations where the state variables are the electric field induction 2-form $\alpha_p = \mathcal{D} \in \Omega^2(Z)$ and the magnetic field induction 2-form $\alpha_q = \mathcal{B} \in \Omega^2(Z)$, the co-energy variables (the variational derivative of the total electromagnetic energy H) are the electric field intensity $\mathcal{E} \in \Omega^1(Z)$ and the magnetic field intensity $\mathcal{H} \in \Omega^1(Z)$. The boundary port variables are the restriction of the electric and magnetic field intensities to the boundary, yielding the energy balance equation

$$\frac{dH}{dt} = \int_{\partial Z} e_\partial \wedge f_\partial = \int_{\partial Z} \mathcal{H} \wedge \mathcal{E} = -\int_{\partial Z} \mathcal{E} \wedge \mathcal{H} \tag{16}$$

with $\mathcal{E} \wedge \mathcal{H}$ is the 2-form corresponding to the *Poynting vector.*

This construction has been generalized to Hamiltonian operators defining Lie-Poisson brackets (in fluid mechanics for instance [2], [7] or corresponding to composition of port Hamiltonian systems [6], [8] as well as to higher-order differential operators [14], [15], the latter being illustrated in subsection III-D of this paper.

B. Port Hamiltonian systems derived from systems of conservation laws on graphs

In this paragraph, we shall present finite-dimensional port Hamiltonian systems which may also be interpreted as systems of conservation laws but defined on some finite-dimensional spatial domain, represented as a graph [3]. The general case, really analogous to the systems of conservation laws presented in the paragraph II-A, defined on k-complexes may be found in [16].

A directed graph $\mathcal{G}$ consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of directed *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. The graph $\mathcal{G}$ is completely specified by its *incidence matrix* $\hat{B}$, which is an $N \times M$ matrix, N being the number of vertices and M being the number of edges, with (i, j)-th element equal to -1 if the j-th edge is an edge towards vertex i, equal to 1 if the j-th edge is an edge originating from vertex i, and 0 otherwise. Its *vertex space* Λ_0 is the vector space of all functions from $\mathcal{V}$ to the linear space of the real numbers $\mathbb{R}$ [2] leading to identify Λ_0 with $\mathbb{R}^N$. Its *edge space* Λ_1 is the vector space of all functions from $\mathcal{E}$ to $\mathbb{R}$, leading to identify Λ_1 with $\mathbb{R}^M$. The dual spaces of Λ_0 and Λ_1 will be denoted by Λ^0, respectively Λ_1. The adjoint map of B is denoted as $B^* : \Lambda^0 \to \Lambda^1$, and is called the *co-incidence* operator and is defined by the transposed matrix $\hat{B}^T$.

We shall consider an *open graph* $\mathcal{G}$, which is obtained from an ordinary graph with set of vertices $\mathcal{V}$ by identifying a subset $\mathcal{V}_b \subset \mathcal{V}$ of N_b *boundary vertices*. The interpretation of $\mathcal{V}_b$ is that these are the vertices that are open to interconnection (i.e., with other open graphs). The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are the N_i *internal vertices* of the open graph.

The splitting of the vertices into internal and boundary vertices induces a splitting of the vertex space and its dual. We will thus write

$$\begin{aligned} \Lambda_0 &= \Lambda_{0i} \oplus \Lambda_{0b} \\ \Lambda^0 &= \Lambda^{0i} \oplus \Lambda^{0b} \end{aligned}$$

where Λ_{0i} is the vertex space corresponding to the internal vertices and Λ_{0b} the vertex space corresponding to the boundary vertices. Consequently, the incidence operator $B : \Lambda_1 \to \Lambda_0$ splits as

$$B = B_i \oplus B_b$$

with $B_i : \Lambda_1 \to \Lambda_{0i}$ and $B_b : \Lambda_1 \to \Lambda_{0b}$.

Furthermore, we will define the *boundary space* Λ_b as the linear space of all functions from the set of boundary vertices $\mathcal{V}_b$ to $\mathbb{R}$ and denote its dual space byΛ^b.

Various Dirac structures may be associated with the incidence relations between the vertex and edges spaces [3], however the applications presented in the section I, concerning electrical circuits, make use of the *Kirchhoff-Dirac structure*, which only involves the flow and effort variables in the edge spaces $\Lambda_1 \times \Lambda^1$ and the boundary spaces $\Lambda_b \times \Lambda^b$ and is defined by

$$\begin{aligned} &\mathcal{D}_K(\mathcal{G}) := \{(f_1, e^1, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ &B_i f_1 = 0, B_b f_1 = f_b, \exists e^{0i} \in \Lambda^{0i} \text{ s.t. } e^1 = -B_i^* e^{0i} - B_b^* e^b\} \end{aligned} \tag{17}$$

These relations are respectively Kirchhoff current law $B_b f_1 = f_b$, (with f_1 and f_b, being the currents through the circuit and at boundary vertices) and Kirchhoff's voltage law $e^1 = -B_i^* e^{0i} - B_b^* e^b$, (with e^1 the voltages across the circuit edges and e^0 and e^b, the potentials at the circuit and boundary vertices).

Consider an LC-circuit, with constitutive relations for a capacitor at edge e are given by

$$\dot{Q}_e = I_e, \quad V_e = \frac{dH_{Ce}}{dQ_e}(Q_e) \tag{18}$$

where Q_e is its charge and $H_{Ce}(Q_e)$ denotes the electric energy stored in the capacitor and the constitutive relations of an inductor

$$\dot{\Phi}_e = V_e, \quad I_e = \frac{dH_{Le}}{d\Phi_e}(\Phi_e) \tag{19}$$

where Φ_e denotes the magnetic flux linkage and $H_{Le}(\Phi_e)$ is the magnetic energy. By identifying the total vector of currents I with $-f_1$, and the vector of voltages V with e^1 the dynamics of the LC-circuit may be expressed as a port-Hamiltonian system

$$\left(-\left(\dot{Q}, \frac{dH_L}{d\Phi}(\Phi)\right), \left(\frac{dH_C}{dQ_e}(Q), \dot{\Phi}\right), f_b, e^b\right) \in \mathcal{D}_K(\mathcal{G})$$

C. Dissipative port-Hamiltonian systems

Dissipative phenomena play a decisive role in physical models in engineering and may be introduced in the frame of port-Hamiltonian systems as closure relations of dissipative type implying part of the port variables. Therefore one splits the port variables into interface variables, denoted by $(f_p, e_p) \in \mathcal{F}_p \times \mathcal{E}_p$ and *dissipative port variables* $(y, z) \in \mathcal{F}_D \times \mathcal{E}_D$.

[2]It may actually be any vector space, for instance the Lie algebra $so(3)$ associated with the rigid body displacements for models of spatial mechanisms [3]

Consider a Dirac structure $\mathcal{D}$ defined on the product space $(\mathcal{F}\times\mathcal{F}_p\times\mathcal{F}_D)\times(\mathcal{E}\times\mathcal{E}_p\times\mathcal{E}_D)$, where $\mathcal{F}$ and $\mathcal{E}$ are vector space endowed with a pairing, and the *dissipative (vector-valued) relation*

$$y = w\,(z)$$

satisfying $\langle z, y\rangle \geq 0$. Then *the dissipative port-Hamiltonian system* is defined by

$$\left(\left(-\frac{\partial x}{\partial t},\, f_p, w\,(z)\right),\, (\delta_x H,\, e_p,\, z)\right) \in \mathcal{D}$$

Examples of such dissipative Hamiltonian systems are given for finite-dimensional RLC circuit in section I, loudspeaker in subection III-A, acoustics in subsection III-B and mechanical system in subsection III-C, and in infinite dimension in subsection III-D.

III. Examples

This section is devoted to an exposition of several applications of port-Hamiltonian framework for the modeling and the simulation of audio systems. Note that sections are independent.

A. Simulation of a non-linear electrodynamic loudspeaker

In this section, the main steps for real time simulation of audio systems are briefly discussed (see [21] for details). We firstly recast a linear model of the loudspeaker in PHS formalism using well known electro-mechanical analogy. Secondly we built a parametric nonlinearity to model the stiffness due to the surround and the spider. Thirdly we design a numerical scheme that preserves passivity for solving the system's dynamic. Results for a *FANE 12-500LS* are given.

A well established electronic analog of the loudspeaker is given figure 3. Choosing the momentum as the state of the total moving mass $x_m = m.v_m$, the kinetic energy is $\mathcal{H}_m(x_m) = \frac{x_m^2}{2.m}$. Newton's law of dynamic is then $\frac{\mathrm{d}x_m}{\mathrm{d}t} = F_m$, and $\frac{\mathrm{d}\mathcal{H}_m(x_m)}{\mathrm{d}x_m} = v_m$. With the analogies $i \leftrightarrow v$ and $e \leftrightarrow F$ (see *eg*. [17], [18]), the mass is equivalent to an inductance. Similarly, choosing the deviation from the equilibrium as the state for the linear stiffness K, the potential energy is $\mathcal{H}_K(x_K) = K\frac{x_K^2}{2}$. which corresponds to a capacitance. Choosing $w_a = v_m$, the damping function is $F_a = z_a(w_a) = R_a.w_a$, which corresponds to a resistance.

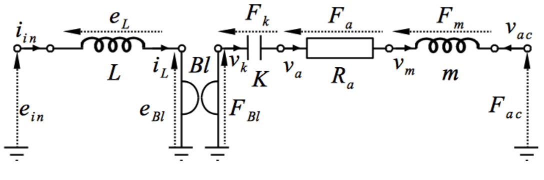


Fig. 3. Basic electric analog of the loudspeaker.

The acoustical port variables are the fluid velocity v_{ac} which is supposed to *stick* at the diaphragm, and the force F_{ac} due to the gradient of pressure between the two sides of the diaphragm. The electro-mechanical conversion (*ie.* Laplace force) is modeled as a conservative gyrator with ratio Bl: $F_{Bl} = Bl.i_L$ and $e_{Bl} = Bl.v_m$. Finally, the loudspeaker can be recasted as the PHS (3) with the definitions of table II.

$\mathbf{x} = \begin{pmatrix} \phi \\ F_m \\ v_m \end{pmatrix}$	$\mathcal{H}(\mathbf{x}) = \mathcal{H}_L(x_1) + \mathcal{H}_m(x_2) + \begin{cases} \mathcal{H}_K(x_3) \\ \text{or} \\ \tilde{\mathcal{H}}_K(x_3) \end{cases}$
$w = v_a$	$F_a = z(w) = R_a\, w$
$\mathbf{J_x} = \begin{pmatrix} 0 & -Bl & 0 \\ +Bl & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 \\ +1 \\ 0 \end{pmatrix}, \mathbf{G_x} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \\ 0 & 0 \end{pmatrix}$ Other matrices are 0 with appropriate dimensions.	

TABLE II
Storage $\mathcal{H}$, dissipation z and structure $\mathbf{S}$ restoring (3) for the loudspeaker of figure 3. The state for the stiffness is $x_K = x_m$. Input/output quantities are $\mathbf{u} := (\nu_1, \nu_2)^T$ and $\mathbf{y} := (i_1, i_2)^T$.

The main non-linearity is the saturation effect of the spider's stiffness (see *eg.* [19], [20]). We built a parametric storage function which phenomenologically restores the saturation, preserving passivity. Here, $\nabla\mathcal{H}_K(x)$ is a linear combination of odd basis functions so that the main part of the system is linear with respect to parameters $(K_0,\, K_{sat})$:

$$\nabla\mathcal{H}_K(x) = K_0 c_0(x) + K_{sat} c_{sat}\left(\frac{x}{x_{sat}}\right) \tag{20}$$

with $c_0(x)$ the linear behaviors around origin and $c_{sat}(x)$ the saturation normalized in $x = \frac{x_{sat}}{2}$. We choose $c_0(x) = x$ and $c_{sat}(x) = \frac{4}{4-\pi}.\left(\tan(\frac{\pi.x}{2}) - \frac{\pi.x}{2}\right)$. The associated energy is

$$\tilde{\mathcal{H}}_K(x) = K_0.\frac{x^2}{2} - \frac{K_{sat}.8.x_{sat}}{\pi(4-\pi)}.\left(\ln\left|\cos\left(\frac{\pi.x}{2x_{sat}}\right)\right| + \frac{\pi^2}{8}.\frac{x^2}{x_{sat}^2}\right).$$

Simply replacing the linear stiffness by its non-linear version in table II leads to a simple non-linear passive model of the loudspeaker for numerical simulations.

Applying the midpoint rule $\delta_t\mathtt{E}(k) = \frac{\mathtt{E}(k+1)-\mathtt{E}(k-1)}{2\delta T}$, with k the sample number and δT the sampling period, a numerical approximation of (3) is $\delta_t\mathtt{E}(k) = P_{\mathcal{S}}(k) - P_{\mathcal{D}}(k)$. Energy $\mathtt{E} = \mathcal{H}(\mathbf{x})$ is a composed function of time. This leads to introduce the discrete Hamiltonian gradient so that

$$\left[\delta_{\mathbf{x}}\mathcal{H}(\mathbf{x},\delta\mathbf{x})\right]_s = \frac{\mathcal{H}_s[x_s(k)+\frac{1}{2}\delta x_s] - \mathcal{H}_s[x_s(k)-\frac{1}{2}\delta x_s]}{\delta x_s}, \tag{21}$$

where $\delta x_s = x_s(k+1) - x_s(k-1)$, $\forall s \in [1\cdots n_S]$. Note that for a set of n_l linear storage components $\mathcal{H}_l = \sum_{s=1}^{n_l} h_s$ with $h_s(.) \equiv \frac{.^2}{2C_s}$, $\delta_{\mathbf{x}_l}\mathcal{H}_l(\mathbf{x}_l) = \mathbf{Q}\mathbf{x}_l$ with $\mathbf{Q} = \mathrm{diag}(\frac{1}{C_1}\cdots\frac{1}{C_{n_l}})$. A numerical scheme preserving the stability for solving equation (3) is given by $\delta_t\mathtt{E}(k) = \left[\delta_{\mathbf{x}}\mathcal{H}(\mathbf{x},\delta\mathbf{x})\right]^T.\delta_t\mathbf{x}(k)$ where $\delta_t\mathbf{x}(k) = \frac{\mathbf{x}(k+1)-\mathbf{x}(k-1)}{2\delta T}$. Hence, placing the sample number in subscript, a numerical version of system (3) preserving the passivity is given by

$$\begin{array}{rcl} \frac{\mathbf{x}_{k+1}-\mathbf{x}_{k-1}}{2\delta T} & = & \mathbf{J_x}.\delta_{\mathbf{x}}\mathcal{H}(\mathbf{x}_k,\delta\mathbf{x}) - \mathbf{K}.\mathbf{z}(\mathbf{w}_k) + \mathbf{G_x}.\mathbf{u}_k, \\ \mathbf{w}_k & = & \mathbf{K}^T.\delta_{\mathbf{x}}\mathcal{H}(\mathbf{x}_k,\delta\mathbf{x}) + \mathbf{J_w}.\mathbf{z}(\mathbf{w}_k) + \mathbf{G_w}.\mathbf{u}_k, \\ \mathbf{y}_k & = & \mathbf{G_x}^T.\delta_{\mathbf{x}}\mathcal{H}(\mathbf{x}_k,\delta\mathbf{x}) + \mathbf{G_w}^T.\mathbf{z}(\mathbf{w}_k) + \mathbf{J}_y.\mathbf{u}_k. \end{array} \tag{22}$$

The simulated loudspeaker is a *FANE* 12-500[3] with m=75g, $L = 2,36$mH, $K = 7,4$kN/m, $R_a = 3.14$kg/s, $Bl = 16,4$N/A, $x_{sat} = 5,17$mm. Simulation are performed in Matlab at the sample rate of 44100kHz. Input are e_{in} a sinusoidal signal at 150Hz with amplitude 100V, and pressure equilibrium F_{ac}=0. First the nonlinearity is discarded setting $K_{\text{sat}} = 0$. Then, the system is made non linear with $K_{\text{sat}} = \frac{K_0}{100}$.

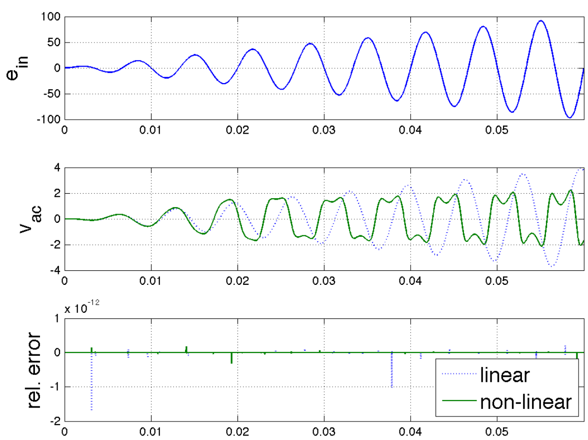


Fig. 4. Input e_{in}(V), output v_{ac}(m.s^{-1}) and relative error on the power balance $(\delta_t \mathtt{E} - P_{\mathcal{S}} + P_{\mathcal{D}})/\delta_t \mathtt{E}$.

We see figure 4 the saturation effect in the non-linear case, which bound the position in $]-x_{sat}, x_{sat}[$. Note that the relative error on the power balance is of two order of magnitude of the machine precision.Simulated electrical impedance is in good accordance with manufacturer's characteristic (see figure 5). Measurement clearly shows a non-standard behavior (*ie.* fractional order integrator). Such refinement can be encompassed through a well established formalism, the so called *diffusive representations* (see [28] for details). This method is designed for real time simulation of guitar amplifiers [21].

B. Simplified model of an air flow modulated by a moving rigid boundary

Here we derive a simple but energetically consistent model of the coupling between an artificial lip (the musician), a jet and an acoustic tube (the instrument), designed to be used in the modeling and control of a brass instruments playing robot.

We consider an irrotationnal 2D flow of an incompressible perfect fluid in a time-varying volume $\Omega(t) = \ell\zeta\xi(t)$, where ℓ and ζ denote the length and the width of the channel, respectively (see figure 6). This flow is contained between a static wall (at bottom) and a mobile wall (at the top), so that the transverse velocities are uniform on these boundaries. For sake of simplicity, the longitudinal velocities are also chosen to be uniform on the left and right boundaries: this assumption makes the corresponding airflows appear as the product of these velocities by the areas of the boundaries.

[3]http://eshop.prodance.cz/Files/FANE_Sovereign_12500LF_Specs.pdf

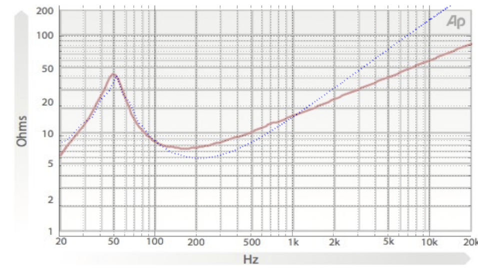


Fig. 5. Electrical impedance: manufacturer (red line), simulated (blue dot).

In [22], it has been shown that the model of the flow can be reduced to an equivalent macroscopic description with averaged variables (the average is taken on the volume for the state variables, and on the control surfaces for the external port variables). We found out that the macroscopic model corresponds to a PHS, described by the differential system in the form (3), with the definitions of table III.

$\mathbf{x}_F = [V_x, V_y, \xi]^T = [\langle v_x\rangle_\Omega, \langle v_y\rangle_\Omega, \xi]^T$	$\mathcal{H}_F(\mathbf{x}_F) := \frac{1}{2}\eta(\xi)\, {V_x}^2 + \frac{1}{2}\eta(\xi)\alpha(\xi)\, {V_y}^2$
$\mathbf{u}_F = \begin{pmatrix} \langle p + \frac{1}{2}\rho\|\mathbf{v}\|^2\rangle_{S_-} \\ \langle p + \frac{1}{2}\rho\|\mathbf{v}\|^2\rangle_{S_+} \\ S_w \langle p\rangle_{S_w} \end{pmatrix} = (P_F^-, P_F^+, F_F)^T$	$\mathbf{y}_F = \begin{pmatrix} S_- v_x(0,t) \\ -S_+ v_x(l,t) \\ -\dot{\xi} \end{pmatrix} = (U_F^-, U_F^+, V_F)^T$
$\mathbf{J_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\eta\alpha} \\ 0 & \frac{2}{\eta\alpha} & 0 \end{pmatrix}, \mathbf{G_x} = \begin{pmatrix} \frac{\zeta\xi}{\eta} & -\frac{\zeta\xi}{\eta} & 0 \\ \frac{\zeta l}{\eta\alpha} & \frac{\zeta l}{\eta\alpha} & -\frac{2}{\eta\alpha} \\ 0 & 0 & 0 \end{pmatrix}$ Other matrices are 0 with appropriate dimensions.	

TABLE III
STORAGE $\mathcal{H}$, INPUTS/OUTPUTS AND STRUCTURE $\mathbf{S}$ OF THE MODEL OF AN AIR FLOW MODULATED BY A MOVING RIGID BOUNDARY. THE SPATIAL AVERAGE OF A QUANTITY $f(x,y,t)$ OVER A SURFACE OR A VOLUME A IS DENOTED BY $\langle f\rangle_A(t) := \frac{1}{A}\int_A f(x,y,t)\,\mathrm{d}A$.

C. Linear finite-dimensional mechanical system, structural damping of Caughey type

Whether by direct modeling or by discretization procedures, the study of the stability of large dimensional systems is a recurrent topic in musical acoustics. Consider an n-d.o.f. finite-dimensional harmonic oscillator, the dynamic equation of which is written in the form [23]:

$$M\,\ddot{q} + C\,\dot{q} + K\,q = 0\,, \tag{23}$$

where $q(t) \in \mathbb{R}^n$ and $M = M^T > 0$, $K = K^T \geq 0$ and the damping matrix is $C = C^T \geq 0$ (that is to say, no gyroscopic term $G\,\dot{q}$ with $G = -G^T$, is taken into account). A PHS description of this system in the form (3) corresponds to that given in table IV.

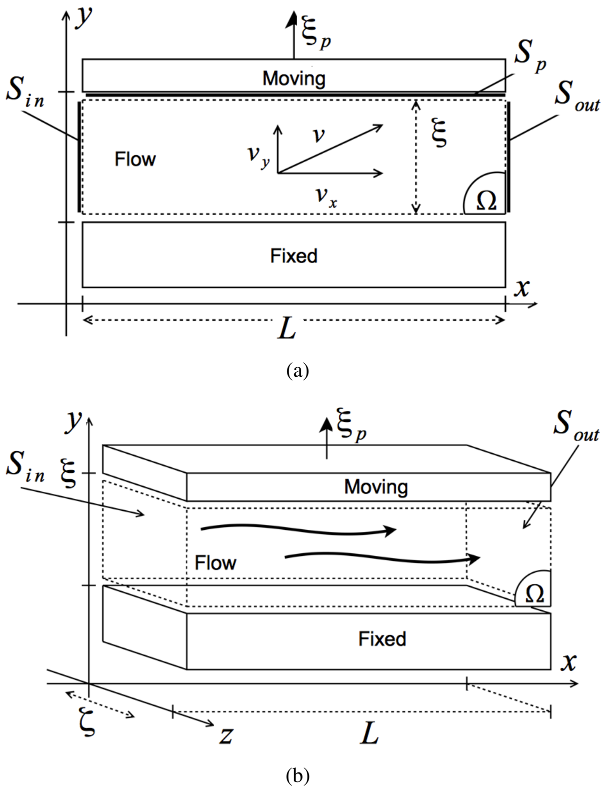


Fig. 6. A 2D irrotationnal flow under a mobile wall (invariant along the z-axis): the velocity field is $\mathbf{v}(x,y,t) = v_x(x,y,t)\mathbf{e}_x + v_y(x,y,t)\mathbf{e}_y$.

$x = \begin{pmatrix} q \\ \partial_t(Mq) \end{pmatrix}$	$H(x) = \frac{1}{2}x^T W x$, with $W = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix}$
$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$	$z(w) = C\,w$
$S = \left(\begin{array}{cc\|c} 0 & I_n & 0 \\ -I_n & 0 & -I_n \\ \hline 0 & I_n & 0 \end{array}\right)$	

TABLE IV
LINEAR FINITE-DIMENSIONAL MECHANICAL SYSTEM WITH LINEAR DAMPING. IN THIS CASE, w REPRESENTS THE VELOCITIES $\partial_t q$, AND $z(w)$ THE DAMPING FORCES.

Following [24], a family of matrices C that leave unchanged the eigen-modes of the conservative system (i.e. with $C = 0$) in (23) is given by

$$C := b_0 M + \sum_{l=1}^{n-1} b_l \, K M^{-1} K \cdots M^{-1} K \,,$$

each term having l occurrences of K and $l-1$ of M^{-1} and where $b_l \geq 0$ (see also [25] for a necessary and sufficient condition). A PHS description of this family is given in [26, lemma 1], which is summarized in table V.

$x = \begin{pmatrix} q \\ \partial_t(Mq) \end{pmatrix}$	$H(x) = \frac{1}{2}x^T W x$, with $W = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix}$
$w = \begin{pmatrix} w_1 \\ \vdots \\ w_{n^2} \end{pmatrix}$	$z(w) = \mathrm{diag}(b_0 I_n, \ldots, b_{n-1} I_n)\, w$
$\mathbf{S} = \left(\begin{array}{cc\|c} 0 & I_n & 0 \\ -I_n & 0 & -G_n \\ \hline 0 & G_n^T & 0 \end{array}\right)$ with $G_n = [M^{1/2}, K^{1/2}, KM^{-1/2}, KM^{-1}K^{1/2}, \ldots]$	

TABLE V
LINEAR FINITE-DIMENSIONAL MECHANICAL SYSTEM WITH DAMPING THAT LEAVES UNCHANGED THE EIGEN-MODES OF THE CONSERVATIVE SYSTEM.

D. Extension to infinite-dimensional systems: example of the Euler Bernoulli beam

In [26, Section 3], the results on dampings in section III-C have been extended to infinite-dimensional systems, for which the function x depends both on time t and space z, and where the matrix S becomes an operator. As an example, the Euler-Bernoulli beam can be described with this formalism as follows (with dimensionless quantities for sake of simplicity).

Denote $u(z,t)$ the transverse displacement of a beam with constant cross-section, with free ends at $z = 0$ and $z = L$, and with Hamiltonian

$$H(x) = \frac{1}{2}\int_0^L ({x_1}^2 + {x_2}^2)\,\mathrm{d}z,$$

where, classically, $x_1 := \partial_z^2 u$ and $x_2 := \partial_t u$. Denoting by δ_x the variational derivative, it comes (see [26, Example 3]),

$$\left(\begin{array}{c} \partial_t x_1 \\ \partial_t x_2 \\ \hline w \end{array}\right) = \mathbf{S} \left(\begin{array}{c} \delta_{x_1} H \\ \delta_{x_2} H \\ \hline z(w) \end{array}\right) \qquad (24)$$

where $\mathbf{S} = \left(\begin{array}{cc|c} 0 & \partial_z^2 & 0 \\ -\partial_z^2 & 0 & -G \\ \hline 0 & G^* & 0 \end{array}\right)$ is formally skew-symmetric, with $G = (1, \partial_z^2)$, and $z(w) = (b_0 w_1, b_1 w_2)^T$; where ($b_0 \geq 0$, $b_1 \geq 0$) ensures the positivity property $\mathbf{z}(\mathbf{w})^T\mathbf{w} \geq 0$ (in this respect, note that other positive maps, even non-linear ones, can be proposed for modelling). In this model, b_0 corresponds to a fluid damping, and b_1 to a structural damping. These parameters are both perceptively relevant for sound synthesis purposes, as previously detailed in [27] (metal, glass or wood mediums).

Remark 1. *In this section, only space dependent damping mechanisms have been dealt with, whereas many time-dependent model can be found in the literature. A widely used family of time-dependent damping models is that with completely monotone kernels, also known as diffusive representations. Among those, one can find the quite popular subclass of fractional integrals and derivatives, with respect to time: all these time-dependent damping models can be recast as a port-Hamiltonian systems, as fully detailed in [28] and references therein.*

IV. Conclusion

This paper has recalled the very general framework of port-Hamiltonian systems, either finite-dimensional or infinite-dimensional, either linear or non-linear, either on classical structures or on graphs, either without or with dissipation. After an introductory example with a simple RLC circuit, many worked-out applications of this powerful formalism to acoustics and electro-acoustics have been presented: an electrodynamic loudspeaker for real time simulation of guitar amplifiers, a simplified model of an air flow modulated by a moving rigid boundary for the modeling and control of a brass instruments playing robot and, last but not least, structural damping models for finite-dimensional mechanical systems and for the Euler-Bernoulli beam, which will be used for real time simulation of an electro-mechanical piano.

References

[1] Duindam, V., Macchelli, A., Stramigioli, S., & Bruyninckx, H. (Ed.) (2009). *Modeling and control of complex physical systems - The port-Hamiltonian Approach*. Springer.

[2] van der Schaft, A. J., & Maschke, B. M. (2002). Hamiltonian formulation of distributed-parameter systems with boundary energy flow. J. of Geometry and Physics, 42(1), 166–194.

[3] van der Schaft, A. J., & Maschke, B. M. (2013). Port-Hamiltonian systems on graphs. SIAM J. on Control and Optimization, 51(2), 906–937.

[4] van der Schaft, A. J., & Maschke, B. M. (1995). The Hamiltonian formulation of energy conserving physical systems with external ports. AEU. Archiv für Elektronik und Übertragungstechnik, 49(5-6), 362–371.

[5] van der Schaft, A. J., & Jeltsema, D. *Port-Hamiltonian Systems Theory: An Introductory Overview*. Foundations and Trends in Systems and Control, 1(2-3):173–378, Now Publisher.

[6] Baaiu, A., Couenne, F., Eberard, D., Jallut, C., Lefèvre, L., Le Gorrec, Y., Maschke, B. & Tayakout-Fayolle, M. (2009). Port-based modelling of mass transport phenomena. Mathematical and Computer Modelling of Dynamical Systems, 15(3), 233–254.

[7] Polner, M. A., & van der Vegt, J. J. W. (2013). A Hamiltonian vorticity-dilatation formulation of the compressible Euler equations. Memorandum 2015, Dept. of Applied Mathematics, University of Twente, Enschede.

[8] Vu, T., Lefèvre, L., & Maschke, B. (2012). Port-hamiltonian formulation for systems of conservation laws: application to plasma dynamics in tokamak reactors. In proc. *4th IFAC workshop on Lagrangian and Hamiltonian Methods for Non Linear Control*, Bertinoro, Italy, pp. 108–113.

[9] Choquet-Bruhat, Y., DeWitt-Morette, C. (1982), *Analysis, Manifolds and Physics*, North-Holland.

[10] Olver, P. J. (2000). *Applications of Lie groups to differential equations*, vol. 107 of *Graduate texts in mathematics*. Springer.

[11] Courant, T. J., & Weinstein, A. (1988). Beyond Poisson structures, In *Séminaire Sud-Rhodanien de Géométrie*, volume 8 of *Travaux en cours*. Hermann.

[12] Courant, T. J. (1990). Dirac manifolds. Trans. of the American Mathematical Society, 319(2), 631–661.

[13] Dorfman, I. Y. (1993). *Dirac structures and integrability of nonlinear evolution equations*. John Wiley.

[14] Le Gorrec, Y., Zwart, H., & Maschke, B. M. (2005). Dirac structures and boundary control systems associated with skew-symmetric differential operators. SIAM J. on Control and Optimization, 44(5), 1864–1892.

[15] Jacob, B., & Zwart, H. J. (2012). *Linear port-Hamiltonian systems on infinite-dimensional spaces*, volume 223 of *Operator Theory: Advances and Applications*. Springer.

[16] van der Schaft, A. J., & Maschke, B. M. (2009). chapter *Conservation laws and lumped system dynamics*, pages 31–48, in *Model-Based Control: Bridging Rigorous Theory and Advanced Technology*, van den Hof, P. M., Scherer, C., & Heuberger, P. S. (Eds). Springer.

[17] Thiele, N. (1971). Loudspeakers in vented boxes: Part 1. Journal of the Audio Engineering Society, 19(5), 382–392.

[18] Thiele, N. (1971). Loudspeakers in vented boxes: Part 2. Journal of the Audio Engineering Society, 19(6), 471–483.

[19] Klippel, W. (1990). Dynamic measurement and interpretation of the nonlinear parameters of electrodynamic loudspeakers. Journal of the Audio Engineering Society, 38(12), 944–955.

[20] Klippel, W. (2005, October). Loudspeaker Nonlinearities - Causes, Parameters, Symptoms. In proc. *119th Audio Engineering Society Convention*, New York, USA, paper 6584.

[21] Falaize-Skrzek, A., & HÃ©lie, T. (2013, October). Simulation of an analog circuit of a wah pedal: a port-Hamiltonian approach. In Audio Engineering Society Convention 135. Audio Engineering Society.

[22] Lopes, N. &. Hélie,T. (2014). Modèle d'interaction Jet/Lèvre préservant le bilan de puissance pour les instruments de type cuivre, in proc. *12ième Congrès Français d'Acoustique* Poitiers, France, pp. 1845–1851.

[23] Géradin, M., & Rixen, D. (1997). Mechanical vibrations. John Wiley.

[24] Caughey, T. K. (1960). Classical normal modes in damped linear dynamic systems. Journal of Applied Mechanics, 27(2), 269–271.

[25] Caughey, T. K., & O'kelly, M. E. J. (1965). Classical normal modes in damped linear dynamic systems. Journal of Applied Mechanics, 32(3), 583–588.

[26] Matignon, D., & Hélie, T. (2013). A class of damping models preserving eigenspaces for linear conservative port-Hamiltonian systems. European Journal of Control, 19(6), 486–494.

[27] Matignon, D., & Hélie, T. (2001). Damping models for the sound synthesis of bar-like instruments. In proc. *5th Conference on Systemics, Cybernetics and Informatics*, Orlando, USA, vol. 10, 541–546.

[28] Le Gorrec, Y., & Matignon, D. (2013). Coupling between hyperbolic and diffusive systems: A port-Hamiltonian formulation. European Journal of Control, 19(6), 505–512